Imperial/TP/2007/OC/03

Spacetime singularity resolution by M-theory fivebranes: calibrated geometry, Anti-de Sitter solutions and special holonomy metrics

Oisín A. P. Mac Conamhna

Theoretical Physics Group, Blackett Laboratory,
Imperial College London, London SW7 2AZ, U.K.

The Institute for Mathematical Sciences,
Imperial College London, London SW7 2PE, U.K.

Abstract

The supergravity description of various configurations of supersymmetric M-fivebranes wrapped on calibrated cycles of special holonomy manifolds is studied. The description is provided by solutions of eleven-dimensional supergravity which interpolate smoothly between a special holonomy manifold and an event horizon with Anti-de Sitter geometry. For known examples of Anti-de Sitter solutions, the associated special holonomy metric is derived. One explicit Anti-de Sitter solution of M-theory is so treated for fivebranes wrapping each of the following cycles: Kähler cycles in Calabi-Yau two-, three- and four-folds; special lagrangian cycles in three- and four-folds; associative three- and co-associative four-cycles in G_2 manifolds; complex lagrangian four-cycles in $Sp(2)$ manifolds; and Cayley four-cycles in $Spin(7)$ manifolds. In each case, the associated special holonomy metric is singular, and is a hyperbolic analogue of a known metric. The analogous known metrics are respectively: Eguchi-Hanson, the resolved conifold and the four-fold resolved conifold; the deformed conifold, and the Stenzel four-fold metric; the Bryant-Salamon-Gibbons-Page-Pope G_2 metrics on an $\mathbb{R}^4$ bundle over S^3, and an $\mathbb{R}^3$ bundle over S^4 or $\mathbb{CP}^2$; the Calabi hyper-Kähler metric on $T^*\mathbb{CP}^2$; and the Bryant-Salamon-Gibbons-Page-Pope $Spin(7)$ metric on an $\mathbb{R}^4$ bundle over S^4. By the AdS/CFT correspondence, a conformal field theory is associated to each of the new singular special holonomy metrics, and defines the quantum gravitational physics of the resolution of their singularities.

1 Introduction

The AdS/CFT correspondence [1] provides a conceptual framework for consistently encoding the geometry of Anti-de Sitter and special holonomy solutions of M-/string theory in a quantum theory. Though the class of spacetimes to which it can be applied is restricted, and unfortunately does not include FLRW cosmologies, it provides the only complete proposal extant for the definition of a quantum theory of gravity. For the prototypical example of $AdS_5 \times S^5/\mathbb{R}^{10}$ and $\mathcal{N} = 4$ super Yang-Mills, the Maldacena conjecture is by now approaching the status of proof [2], [3]. The literature on the correspondence is enormous, from applications in pure mathematics to phenomenological investigations. On the phenomenological front, much effort has been devoted to extending the AdS/CFT correspondence from $\mathcal{N} = 4$ super Yang-Mills to more realistic field theories [4] and even QCD itself [5], [6]. Also, recent developments have raised the hope that we may soon be able to use AdS/CFT to test M-/string theory in the lab [7]-[10]. On the mathematical front, the motivation provided by the AdS/CFT correspondence has stimulated spectacular progress in differential geometry; early work on the correspondence showed that there is a deep interplay between Anti-de Sitter solutions of M-/string theory, singular special holonomy manifolds and conformal field theories [11], [12]. This relationship has since been the topic of intense investigation; a recent highlight has been the beautiful work on Sasaki-Einstein geometry, toric Calabi-Yau three-folds and the associated conformal field theories [13]-[19]. What has become clear is that the geometry of a supersymmetric AdS/CFT dual involves an Anti-de Sitter manifold, a singular special holonomy manifold[1] and a supergravity solution which, in a sense that will be made more precise, interpolates smoothly between them. This geometrical relationship, between Anti-de Sitter manifolds and singular special holonomy manifolds, in the context of the AdS/CFT correspondence in M-theory, is the subject of this paper.

The canonical example of this relationship, from IIB, is that between conically singular Calabi-Yau three-folds and Sasaki-Einstein AdS_5 solutions of IIB supergravity. Each of these geometries, individually, is a supersymmetric solution of IIB, preserving eight supercharges. Furthermore, the manifolds may be superimposed[2] to obtain another supersymmetric solution of IIB, admitting four supersymmetries. This interpolating solution - the supergravity description of D3 branes at a conical Calabi-Yau singularity - has metric

$$\mathrm{d}s^2 = \left(\mathrm{A} + \frac{\mathrm{B}}{r^4}\right)^{-1/2} \mathrm{d}s^2(\mathbb{R}^{1,3}) + \left(\mathrm{A} + \frac{\mathrm{B}}{r^4}\right)^{1/2} \Big(\mathrm{d}r^2 + r^2 \mathrm{d}s^2(\mathrm{SE}_5)\Big), \tag{1.1}$$

for constants A, B and a Sasaki-Einstein five-metric $\mathrm{d}s^2(\mathrm{SE}_5)$. Setting B=0 gives the IIB solution $\mathbb{R}^{1,3} \times \mathrm{CY}_3$, while setting A=0 gives the solution $\mathrm{AdS}_5 \times \mathrm{SE}_5$. For positive A, B, the solution

[1] With the obviously special non-singular exception of flat space.

[2] Because, with a suitable ansatz including both, the supergravity field equations linearise.

is globally smooth, and contains two distinct asymptotic regions: a spacelike infinity where the metric asymptotes to that of the Calabi-Yau, and an internal spacelike infinity, where the metric asymptotes to that of the Anti-de Sitter, on an event horizon at infinite proper distance. The causal structure of these solutions is discussed in detail in [20]. The Calabi-Yau singularity is excised in the interpolating solution, and removed to infinity; an important feature of the interpolating solution is that it admits a globally-defined $SU(3)$ structure.

The AdS/CFT correspondence tells us how to perform this geometrical interpolation in a quantum framework. Open string theory on the singular Calabi-Yau reduces, at low energies, to a conformally invariant quiver gauge theory, at weak 't Hooft coupling. This is the low-energy effective field theory on the world-volume of a stack of probe D3 branes located at the singularity. The gauge theory encodes the toric data of the Calabi-Yau. The same quiver gauge theory, at strong 't Hooft coupling, is identical to IIB string theory on the $\mathrm{AdS}_5 \times \mathrm{SE}_5$; by the AdS/CFT dictionary, the CFT also encodes the Sasaki-Einstein data of the AdS solution. Clearly, it can only do this for both the Calabi-Yau and the AdS_5 if their geometry is intimately related. In the classical regime, this relationship is provided by the interpolating solution. In the quantum regime, the relationship is provided by the CFT itself; the interpolation parameter is the 't Hooft coupling. In effect, the CFT is telling us how to cut out the Calabi-Yau singularity quantum gravitationally, and replace it with an event horizon with the geometry of Anti-de Sitter.

The correspondence is best understood for branes at conical singularities of special holonomy manifolds. However, starting from the work of Maldacena and Nuñez [21], many supersymmetric AdS solutions of M-/string theory have been discovered,[22]-[29], [13], which cannot be interpreted as coming from a stack of branes at a conical singularity. Instead, they have been interpreted as the near-horizon limits of the supergravity description of branes wrapped on calibrated cyles of special holonomy manifolds. The CFT dual of the AdS/special holonomy manifolds is the low-energy effective theory on the unwrapped worldvolume directions of the branes. A brane, heuristically envisioned as a hypersurface in spacetime, can wrap a calibrated cycle in a special holonomy manifold, while preserving supersymmetry. A heuristic physical argument as to why this is possible is that a calibrated cycle is volume-minimising in its homology class; as a probe brane has a tension, it will always try to contract, and so a wrapped probe brane is only stable if it wraps a minimal cycle. The supergravity description of a stack of wrapped branes, by analogy with that of branes at conical singularities, should be a supergravity solution which smoothly interpolates between a special holonomy manifold with an appropriate calibrated cycle, and an event horizon with Anti-de Sitter geometry. As the notion of an interpolating solution is central to this paper, a more careful definition of what is meant by these words will now be given.

Definition 1 Let $\mathcal{M}_{AdS}$ be a d-dimensional manifold admiting a warped-product AdS metric g_{AdS}, that, together with a matter content F_{AdS}, gives a supersymmetric solution of a supergravity theory in d dimensions. Let $\mathcal{M}_{SH}$ be a d-dimensional manifold admitting a special holonomy metric g_{SH}, which gives a supersymmetric vacuum solution of the supergravity with holonomy $G \subset Spin(d-1)$. Let $\mathcal{M}_I$ be a d-dimensional manifold admitting a globally-defined G-structure, together with a metric g_I and a matter content F_I that give a supersymmetric solution of the supergravity. Then we say that $(\mathcal{M}_I, \mathrm{g}_I, \mathrm{F}_I)$ is an *interpolating solution* if for all $\epsilon, \zeta > 0$, there exist open sets $O_{AdS} \subset \mathcal{M}_{AdS}$, $O_I, O'_I \subset \mathcal{M}_I$, $O_{SH} \subset \mathcal{M}_{SH}$, such that for all points $p_{AdS} \in O_{AdS}$, $p_I \in O_I$, $p'_I \in O'_I$, $p_{SH} \in O_{SH}$,

$$|\mathrm{g}_{AdS}(p_{AdS}) - \mathrm{g}_I(p_I)| < \epsilon, \quad |\mathrm{g}_{SH}(p_{SH}) - \mathrm{g}_I(p'_I)| < \zeta. \tag{1.2}$$

We also define the following useful pieces of vocabulary:

Definition 2 If for a given pair $(\mathcal{M}_{AdS}, \mathrm{g}_{AdS}, \mathrm{F}_{AdS})$, $(\mathcal{M}_{SH}, \mathrm{g}_{SH}, \mathrm{F}_{SH})$, there exists an interpolating solution, then we say that $\mathcal{M}_{SH}$ is a *special holonomy interpolation* of $\mathcal{M}_{AdS}$ and that $\mathcal{M}_{AdS}$ is an *Anti-de Sitter interpolation* of $\mathcal{M}_{SH}$. Collectively, we refer to $(\mathcal{M}_{AdS}, \mathrm{g}_{AdS}, \mathrm{F}_{AdS})$ and $(\mathcal{M}_{SH}, \mathrm{g}_{SH}, \mathrm{F}_{SH})$ as an *interpolating pair*.

The objective of this paper is to derive candidate special holonomy interpolations of some of the wrapped fivebrane near-horizon limit AdS solutions of [22]-[25]. In [31], candidate special holonomy interpolations of the AdS_5 M-theory solutions of [21] were derived. These AdS solutions describe the near-horizon limit of fivebranes wrapped on Kähler two-cycles in Calabi-Yau two-folds and three-folds. As these results fit nicely into the more extensive picture presented here, they will be reviewed briefly below. The new special holonomy metrics that will be derived here are candidate interpolations of: the AdS_3 solution of [24], describing the near-horizon limit of fivebranes wrapped on a Kähler four-cycle in a four-fold; the AdS_4 solution of [23], interpreted in [24] as the near-horizon limit of fivebranes on a special lagrangian (SLAG) three-cycle in a three-fold; the AdS_3 solution of [24], for fivebranes on a SLAG four-cycle in a four-fold; the AdS_4 solution of [22], for fivebranes on an associative three-cycle in a G_2 manifold; the AdS_3 solution of [24], for fivebranes on a co-associative four-cycle in a G_2 manifold; the AdS_3 solution of [25], for fivebranes on a complex lagrangian (CLAG) four-cycle in an $Sp(2)$ manifold; and the AdS_3 solution of [24], for fivebranes on a Cayley four-cycle in a $Spin(7)$ manifold. This paper therefore provides one candidate interpolating pair for every type of cycle on which M-theory fivebranes can wrap, in all manifolds of dimension less than ten with irreducible holonomy, with the exception of Kähler four-cycles in three-folds and quaternionic Kähler four-cycles in $Sp(2)$ manifolds, for which no AdS solutions are known to the author.

No interpolating solutions of eleven-dimensional supergravity which describe wrapped branes are known. However, based on various symmetry and supersymmetry arguments, the differential equations they satisfy are known, for all types of calibrated cycles in all special holonomy manifolds that play a rôle in M-theory. These equations will be called the wrapped brane equations; there is an extensive literature on their derivation [32]-[41]; the most general results are those of [39]-[41]. The key point that will be exploited here is that *both* members of an interpolating pair should individually be a solution of the wrapped brane equations, with a suitable ansatz for the interpolating solution. This is just like what happens for an interpolating solution associated to a conical special holonomy manifold.

One of the many important results of [13] was to show how any AdS_5 solution of M-theory, coming from fivebranes on a Kähler two-cycle in a three-fold, satisfies the appropriate wrapped brane equations. The canonical frame of the AdS_5 solutions, defined by their eight Killing spinors, admits an $SU(2)$ structure. The AdS_5 solutions may also be re-written in such a way that the canonical AdS_5 frame is obscured, but a canonical $\mathbb{R}^{1,3}$ frame is made manifest. This frame admits an $SU(3)$ structure, and is defined by *half* the Killing spinors of the AdS_5 solution. And it is this Minkowski $SU(3)$ structure which satisfies the wrapped brane equations. By definition, any interpolating solution describing fivebranes on a Kähler two-cycle in a three-fold admits a globally-defined $SU(3)$ structure; this structure smoothly matches on to the $SU(3)$ structure of the Calabi-Yau and also to the canonical $SU(3)$ structure of the AdS_5 solution. This construction has since been systematically extended to all calibrated cycles in manifolds with irreducible holonomy of relevance to M-theory in [39], [40], [41], and, starting from the wrapped brane equations, has been used to classify (ie, derive the differential equation satisfied by) all supersymmetric AdS solutions of M-theory which have a wrapped-brane origin.

The strategy used here to construct candidate special holonomy interpolations of the AdS solutions is therefore the following. We first construct the canonical Minkowski frames and structures of the AdS solutions, which satisfy the appropriate wrapped brane equations. We then use these as a guide to formulating a suitable ansatz for an interpolating solution. It is then a (reasonably) straightforward matter to determine the most general special holonomy solution of the AdS-inspired ansatz for the interpolating solution. In each case, the special holonomy metric thus obtained is the proposed interpolation of the AdS solution. No attempt has been made to determine the interpolating solutions themselves. It is therefore a matter of conjecture whether the special holonomy metrics obtained are indeed interpolations of the AdS solutions. However the results are sufficiently striking that it is reasonable to believe that for the proposed interpolating pairs an interpolating solution does indeed exist.

As an illustration of this procedure, consider the results of [31] for the proposed interpolation of the $\mathcal{N}=2$ AdS_5 solution of [21], describing the near-horizon limit of fivebranes on a Kähler

two-cycle in a two-fold. When re-written in the canonical Minkowski frame, the AdS solution is of the form

$$\mathrm{d}s^2 \;=\; L^{-1}\Big[\mathrm{d}s^2(\mathbb{R}^{1,3})+\frac{F}{2}\mathrm{d}s^2(H^2)\Big]+L^2\Big[F^{-1}\Big(\mathrm{d}u^2+u^2(\mathrm{d}\psi-P)^2\Big)+\mathrm{d}t^2+t^2\mathrm{d}s^2(S^2)\Big], \tag{1.3}$$

where[3] $\mathrm{d}P=\mathrm{Vol}[H^2]$, the period of ψ is 2π and F, L are known functions of the coordinates u and t. The ansatz for the interpolating solution is then simply that F, L are allowed to be arbitrary functions of u, t. The most general special holonomy solution with this ansatz is

$$\mathrm{d}s^2=\mathrm{d}s^2(\mathbb{R}^{1,6})+\mathrm{d}s^2(\mathcal{N}_\tau), \tag{1.4}$$

where, up to an overall scale,

$$\mathrm{d}s^2(\mathcal{N}_\tau)=\frac{R^2}{4}\left[\mathrm{d}s^2(H^2)+\left(\frac{1}{R^4}-1\right)(\mathrm{d}\psi-P)^2\right]+\left(\frac{1}{R^4}-1\right)^{-1}\mathrm{d}R^2. \tag{1.5}$$

The range of R is $R\in(0,1]$. At $R=1$, an S^2 degenerates smoothly, and a H^2 bolt stabilises. At $R=0$, the metric is singular, where the Kähler H^2 cycle degenerates. In the probe-brane picture, the fivebranes should be thought of as wrapping the H^2 at the singularity. Otherwise, they can always decrease their worldvolume by moving to smaller R. This incomplete special holonomy metric is to be compared with the Eguchi-Hanson metric [42], which is

$$\mathrm{d}s^2(\mathrm{EH})=\frac{R^2}{4}\left[\mathrm{d}s^2(S^2)+\left(1-\frac{1}{R^4}\right)(\mathrm{d}\psi-P)^2\right]+\left(1-\frac{1}{R^4}\right)^{-1}\mathrm{d}R^2, \tag{1.6}$$

where now $\mathrm{d}P=\mathrm{Vol}[S^2]$. As is well known, this metric is complete in the range $R\in[1,\infty)$. At $R=1$, an S^2 degenerates smoothly and a Kähler S^2 bolt stabilises.

In every case, the conjectured special holonomy interpolations of the AdS solutions derived in this paper are singular, and they have exactly the same relationship with known complete special holonomy metrics as that of (1.5) with Eguchi-Hanson. To make the pattern clear, it worth quoting one more example now. The conjectured special holonomy interpolation of the AdS_3 solution of [24] for fivebranes on a Cayley four-cycle in a $Spin(7)$ manifold is

$$\mathrm{d}s^2=\mathrm{d}s^2(\mathbb{R}^{1,2})+\mathrm{d}s^2(\mathcal{N}_\tau), \tag{1.7}$$

where, up to an overall scale,

$$\mathrm{d}s^2(\mathcal{N}_\tau)=\frac{9}{20}R^2\mathrm{d}s^2(H^4)+\frac{36}{100}R^2\left(\frac{1}{R^{10/3}}-1\right)\mathrm{D}Y^a\mathrm{D}Y^a+\left(\frac{1}{R^{10/3}}-1\right)^{-1}\mathrm{d}R^2, \tag{1.8}$$

[3]Here, and throughout, $\mathrm{d}s^2(AdS_n)$, $\mathrm{d}s^2(H^n)$, $\mathrm{d}s^2(S^n)$, denote the maximally symmetric Einstein metrics on n-dimensional AdS manifolds, n-hyperboloids or n-spheres with unit radius of curvature, respectively. The cartesian metric on flat space will be denoted by $\mathrm{d}s^2(\mathbb{R}^n)$. The volume form on a unit n-hyperboloid or n-sphere will be denoted by $\mathrm{Vol}[H^n]$, $\mathrm{Vol}[S^n]$, respectively.

where the Y^a are constrained coordinates on an S^3 and D will be defined later. The range of R is $R \in (0,1]$; at $R=1$ the S^3 degenerates smoothly and a H^4 bolt stabilises. At $R=0$ the metric is singular where the H^4 Cayley four-cycle degenerates. This metric is to be compared with the $Spin(7)$ metric on an $\mathbb{R}^4$ bundle over S^4, first found by Bryant and Salamon [43] and later independently by Gibbons, Page and Pope [44]:

$$\mathrm{d}s^2(\mathrm{BSGPP}) = \frac{9}{20}R^2\mathrm{d}s^2(S^4) + \frac{36}{100}R^2\left(1-\frac{1}{R^{10/3}}\right)\mathrm{D}Y^a\mathrm{D}Y^a + \left(1-\frac{1}{R^{10/3}}\right)^{-1}\mathrm{d}R^2, \qquad (1.9)$$

This metric is complete in the range $R \in [1,\infty)$; at $R=1$ an S^4 degenerates smoothly and a Cayley S^4 bolt stabilises.

This relationship with known complete special holonomy metrics is a universal feature of all the proposed special holonomy interpolations of this paper. As this series of incomplete special holonomy metrics has so many features in common, they will be given a collective name, the $\mathcal{N}_\tau$ *series*. Though they have been derived here from the AdS M-theory solutions *ab initio*, they may be obtained in a much simpler way *a posteriori*, by analytic continuation of known complete metrics[4]. In every case, they may be obtained from a known complete metric with a radial coordinate of semi-infinite range, at the endpoint of which an S^m degenerates and a calibrated S^n (or, as appropriate, $\mathbb{CP}^2$) cycle stabilises. The $\mathcal{N}_\tau$ series is obtained by changing the sign of the scalar curvature of the bolt and analytically continuing the dependence of the metric on the radial coordinate. This generates a special holonomy metric with a "radial" coordinate of finite range, with a smoothly degenerating S^m and a stabilised H^n (or Bergman) bolt at one endpoint, and a singular degeneration at the other. For the Calabi-Yau $\mathcal{N}_\tau$ with Kähler cycles in three-folds and four-folds, the analogous known metrics are the resolved conifold of [45], [46], and its four-fold analogue (see [47] for useful additional background on the resolved conifold). For the Calabi-Yau $\mathcal{N}_\tau$ with SLAG cycles, the analogous known metrics are the Stenzel metrics [48] (see [49], [50] for useful background on the Stenzel metrics). The Stenzel two-fold metric coincides with Eguchi-Hanson, and the Stenzel three-fold metric coincides with the deformed conifold metric of [45] (see [51], [47] for additional background on the deformed conifold). For the G_2 $\mathcal{N}_\tau$ metrics with co-associative cycles, the analogous known metrics are the BSGPP metrics [43], [44] on $\mathbb{R}^3$ bundles over S^4 or $\mathbb{CP}^2$. For the G_2 $\mathcal{N}_\tau$ metric with an associative cycle, the analogous known metric is the BSGPP metric [43], [44] on an $\mathbb{R}^4$ bundle over S^3. See [52], [53], [50] for more background on the complete G_2 metrics. For the $Sp(2)$ $\mathcal{N}_\tau$ metric with a CLAG cycle, the analogous known metric is the Calabi metric on $T^*\mathbb{CP}^2$ [54]; the Calabi metric is the unique complete regular hyper-Kähler eight-manifold of

[4]The $\mathcal{N}_\tau$ metrics have almost certainly been found before, though because they are incomplete, they have been presumably been rejected hitherto as pathological and uninteresting. What now makes them interesting is their interpretation as special holonomy interpolations of AdS solutions, for which their incompleteness is probably a pre-requisite: see conjecture 2 below.

co-homogeneity one [55]; for further background on the Calabi metric, see [56]. Finally, for the $Spin(7)$ $\mathcal{N}_\tau$ metric with a Cayley four-cycle, we have seen that the analogous known metric is the BSGPP metric on an $\mathbb{R}^4$ bundle over S^4; see [52], [53], [50] for more details.

What is most striking about the conjectured special holonomy interpolations obtained here is that they are all singular. As occurs in the conical context, the expectation is that the singularity of the special holonomy manifold is excised in the interpolating solution, and that the conformal dual of the geometry gives a quantum gravitational definition of this process. If this is correct, then a singularity of the special holonomy manifold is an essential ingredient of the geometry of AdS/CFT. It would also explain a hitherto rather puzzling feature of the AdS solutions studied here, all of which were originally constructed in gauged supergravity. While for the $\mathcal{N}_\tau$ series it is possible to obtain the known special holonomy manifolds by replacing the H^n factors with S^n factors, for their AdS interpolations this does not seem to be possible; the AdS solutions exist only for hyperbolic cycles. This makes sense if an AdS/CFT dual can exist only for a singular special holonomy manifold; otherwise, if AdS solutions like those studied here, but with S^n cycles, existed, their special holonomy interpolations would be non-singular. Another way of saying this is that it seems that a conformal field theory can be associated to the singular $\mathcal{N}_\tau$ series of special holonomy metrics, but *not* to their non-singular known analogues. If this idea is correct, it means that what the AdS/CFT correspondence is ultimately describing is the quantum gravity of singularity resolution for special holonomy manifolds. We formalise the geometry of this idea in the following two conjectures.

Conjecture 1 Every supersymmetric Anti-de Sitter solution of M-/string theory admits a special holonomy interpolation.

Conjecture 2 With the exception of flat space, the metric on every special holonomy manifold admitting an Anti-de Sitter interpolation is incomplete.

The organisation of the remainder of this paper is as follows. In section two, as useful introductory material, we will review the relationship between the canonical AdS and Minkowski frames for AdS solutions, how to pass from one to the other by means of a frame rotation, and the relationship between the AdS and wrapped brane structures. In section three, we will derive the conjectured special holonomy interpolations of AdS solutions for fivebranes wrapped on cycles in Calabi-Yau manifolds. Section four is devoted to the proposed $Sp(2)$ interpolating pair, section five to the G_2 interpolating pairs and section six to the $Spin(7)$ interpolating pair. In section seven we conclude and discuss interesting future directions.

2 Canonical Minkowski frames for AdS manifolds

In this section we will review how the canonical AdS frame defined by all the Killing spinors of a supersymmetric AdS solution is related to its canonical Minkowski frame defined by half its Killing spinors; for more details, the reader is referred to [13], [39]-[41]. The canonical Minkowski structure of an AdS solution is the one which can match on to the G-structure of an interpolating solution. This phenomenon - the matching of the structure defined by half the supersymmetries of the AdS manifold to that of an interpolating solution - is another, more precise way of stating the familiar feature of supersymmetry doubling in the near-horizon limit of a supergravity brane solution.

We will in fact distinguish two cases, which will be discussed seperately. The AdS solutions we study for fivebranes on cycles in manifolds of $SU(2)$, $SU(3)$ or G_2 holonomy have purely magnetic fluxes. This means that no membranes are present in the geometry. However, the AdS solutions for fivebranes on four-cycles in eight-manifolds ($Spin(7)$, $SU(4)$ or $Sp(2)$ holonomies) have both electric and magnetic fluxes. In probe-brane language, we can think of a stack of fivebranes wrapped a four-cycle in the eight-manifold. We also have a stack of membranes extended in the three overall transverse directions to the eight-manifold. The membrane stack intersects the fivebrane stack in a string; the low-energy effective field theory on the string worldvolume is then the two-dimensional dual of the AdS_3 solutions that come from these geometries. The presence of the membranes complicates the relationship of the AdS and Minkowski frames a little, so first we will discuss the case of fivebranes alone, and purely magnetic fluxes.

2.1 AdS spacetimes from fivebranes on cycles in $SU(2)$, $SU(3)$ and G_2 manifolds

The metric of an interpolating solution describing a stack of fivebranes wrapped on a calibrated cycle in a Calabi-Yau two- or three-fold, or a G_2 manifold, takes the form

$$\mathrm{d}s^2 = L^{-1}\mathrm{d}s^2(\mathbb{R}^{1,p}) + \mathrm{d}s^2(\mathcal{M}_q) + L^2\Big(\mathrm{d}t^2 + t^2\mathrm{d}s^2\left(S^{10-p-q}\right)\Big), \tag{2.1}$$

where $\mathcal{M}_q$ admits a globally-defined $SU(2)$, $SU(3)$ or G_2 structure respectively. The Minkowski isometries are isometries of the full solution, and the flux has no components along the Minkowski directions. The dimensionality of $\mathcal{M}_q$ is $q = 4, 6, 7$, respectively. The dimensionality of the unwrapped fivebrane worldvolume is $p + 1$, so $p = 3$ for a Kähler two-cycle, $p = 2$ for a SLAG or associative three-cycle, and $p = 1$ for a co-associative four-cyle. The intrinsic torsion of the G-structure on $\mathcal{M}_q$ must satisfy certain conditions, implied by supersymmetry and the four-form Bianchi identity. These conditions are what are called the wrapped brane equations; they will be

given for each case below, and need not concern us now. For more details, the reader is referred to [39].

Our interest here is how to obtain a warped product AdS metric from the wrapped-brane metric (2.1), and vice versa. The first step is to recognise that every warped-product AdS_{p+2} metric, written in Poincaré coordinates, may be thought of as a special case of a warped $\mathbb{R}^{1,p}$ metric. If the AdS warp factor is denoted by λ, and is independent of the AdS coordinates, then

$$\lambda^{-1}\mathrm{d}s^2(AdS_{p+2}) = \lambda^{-1}[e^{-2r}\mathrm{d}s^2(\mathbb{R}^{1,p}) + \mathrm{d}r^2]. \tag{2.2}$$

Therefore our first step is to identify $L = \lambda e^{2r}$ in (2.1), with r the AdS radial coordinate. The next step is to pick out the AdS radial direction $\hat{r} = \lambda^{-1/2}\mathrm{d}r$ from the space transverse to the $\mathbb{R}^{1,p}$ factor in (2.1). In the cases of interest to us, the AdS radial direction is a linear combination of the radial direction $\hat{v} = L\mathrm{d}t$ on the overall transverse space, and a radial direction in $\mathcal{M}_q$, transverse to the wrapped cycle. We denote this radial basis one-form on $\mathcal{M}_q$ by $\hat{u}$. Thus we can obtain the AdS radial basis one-form by a local rotation of the frame of (2.1):

$$\hat{r} = \sin\theta\hat{u} + \cos\theta\hat{v}, \tag{2.3}$$

for some local angle θ which we take to be independent of r. Denoting the orthogonal linear combination in the AdS frame by $\hat{\rho}$, we have

$$\hat{\rho} = \cos\theta\hat{u} - \sin\theta\hat{v}. \tag{2.4}$$

Now, imposing closure of $\mathrm{d}t$ and r-independence of θ, we get

$$\hat{\rho} = \frac{\lambda}{2\sin\theta}\mathrm{d}(\lambda^{-3/2}\cos\theta). \tag{2.5}$$

Defining a coordinate ρ for the AdS frame according to $\rho = \lambda^{-3/2}\cos\theta$, we get

$$\begin{aligned} t &= -\frac{\rho}{2}e^{-2r}, \\ \hat{\rho} &= \frac{\lambda}{2\sqrt{1-\lambda^3\rho^2}}\mathrm{d}\rho. \end{aligned} \tag{2.6}$$

Finally, we impose that the metric on the space tranverse to the AdS factor is independent of the AdS radial coordinate, and (in deriving the AdS supersymmetry conditions from the wrapped brane equations) that the flux has no components along the AdS radial direction. Thus we obtain the (for our purposes) general AdS_{p+2} metric contained in (2.1):

$$\mathrm{d}s^2 = \lambda^{-1}\left[\mathrm{d}s^2(AdS_{p+2}) + \frac{\lambda^3}{4}\left(\frac{\mathrm{d}\rho^2}{1-\lambda^3\rho^2} + \rho^2\mathrm{d}s^2\left(S^{10-p-q}\right)\right)\right] + \mathrm{d}s^2(\mathcal{M}_{q-1}), \tag{2.7}$$

where $\mathrm{d}s^2(\mathcal{M}_{q-1})$ is defined by

$$\mathrm{d}s^2(\mathcal{M}_q) = \mathrm{d}s^2(\mathcal{M}_{q-1}) + \hat{u}\otimes\hat{u}. \tag{2.8}$$

In addition, we have

$$\hat{u} = \lambda\left(\sqrt{\frac{1-\lambda^3\rho^2}{\lambda^3}}\mathrm{d}r + \sqrt{\frac{\lambda^3}{1-\lambda^3\rho^2}}\frac{\rho}{2}\mathrm{d}\rho\right). \tag{2.9}$$

Since in general we know the relationship between the Minkowski-frame coordinate t and the AdS frame coordinates r,ρ, when we know λ explicitly for a particular solution, we can integrate (2.9) to find an explicit coordinatisation of the AdS solution in the Minkowski frame. Thus we can pass freely from one frame to the other, for any explicit solution.

Having discussed the relationship of the frames, let us now discuss the relationship between the structures. Since, in passing from (2.1) to (2.7) we pick out a preferred direction on $\mathcal{M}_q$, the G-structure of (2.1) on $\mathcal{M}_q$ is reduced to a G' structure on $\mathcal{M}_{q-1}$ in (2.7). For $q=4$, the $SU(2)$ structure on $\mathcal{M}_4$ is reduced to an identity structure on $\mathcal{M}_3$; the $SU(2)$ forms on $\mathcal{M}_4$ decompose according to

$$\begin{aligned} J_4 &= e^1\wedge e^2 + e^3\wedge\hat{u}, &(2.10)\\ \Omega_4 &= (e^1+ie^2)\wedge(e^3+i\hat{u}), &(2.11)\end{aligned}$$

with

$$\mathrm{d}s^2(\mathcal{M}_4) = e^1\otimes e^1 + e^2\otimes e^2 + e^3\otimes e^3 + \hat{u}\otimes\hat{u}. \tag{2.12}$$

For $q=6$, the $SU(3)$ structure on $\mathcal{M}_6$ reduces to an $SU(2)$ structure on $\mathcal{M}_5$; the $SU(3)$ structure forms decompose according to

$$\begin{aligned} J_6 &= J_4 + e^5\wedge\hat{u},\\ \Omega_6 &= \Omega_4\wedge(e^5+i\hat{u}), \end{aligned} \tag{2.13}$$

with

$$\mathrm{d}s^2(\mathcal{M}_6) = \mathrm{d}s^2(\mathcal{M}_5) + \hat{u}\otimes\hat{u} = \mathrm{d}s^2(\mathcal{M}_4) + e^5\otimes e^5 + \hat{u}\otimes\hat{u}, \tag{2.14}$$

and the $SU(2)$ structure of the AdS frame is defined on $\mathcal{M}_4$. For $q=7$, the G_2 structure on $\mathcal{M}_7$ reduces to an $SU(3)$ structure on $\mathcal{M}_6$; the G_2 structure forms decompose according to

$$\begin{aligned} \Phi &= J_6\wedge\hat{u} - \mathrm{Im}\Omega_6,\\ \Upsilon &= \frac{1}{2}J_6\wedge J_6 + \mathrm{Re}\Omega_6\wedge\hat{u}, \end{aligned} \tag{2.15}$$

with

$$\mathrm{d}s^2(\mathcal{M}_7) = \mathrm{d}s^2(\mathcal{M}_6) + \hat{u} \otimes \hat{u}, \tag{2.16}$$

and the $SU(3)$ structure of the AdS frame is defined on $\mathcal{M}_6$.

2.2 AdS spacetimes from fivebranes on four-cycles in eight-manifolds of Spin(7), SU(4) or Sp(2) holonomy

As discussed above, because of the presence of non-zero electric flux for AdS_3 solutions from fivebranes on four-cycles in eight-manifolds, the relationship between the canonical AdS and Minkowski frames of the AdS solutions is a little more complicated. These systems are the subject of [41], to which the reader is referred for more details[5]. The metric of an interpolating solution describing a stack of fivebranes wrapped on a four-cycle in an eight-manifold, with a stack of membranes extended in the transverse directions, takes the form

$$\mathrm{d}s^2 = L^{-1}\mathrm{d}s^2(\mathbb{R}^{1,1}) + \mathrm{d}s^2(\mathcal{M}_8) + C^2\mathrm{d}t^2. \tag{2.17}$$

Again, the Minkowski isometries are isometries of the full solution, the electric flux contains a factor proportional to the Minkowski volume form, and the magnetic flux has no components along the Minkowski directions. The Minkowski directions represent the unwrapped fivebrane worldvolume directions; the membranes extend in these directions and also along $\mathrm{d}t$. Note that in this case the warp factor of the overall transverse space (the $\mathbb{R}$ coordinatised by t) is independent of the Minkowski warp factor. The global G-structure is defined on $\mathcal{M}_8$; the structure group is $Spin(7)$, $SU(4)$ or $Sp(2)$, as appropriate. Again, supersymmetry, the four-form Bianchi identity, and now, the four-form field equation imply restrictions on the intrinsic torsion of the global G-structure. These equations, the wrapped brane equations for these systems, are given in [41].

To obtain an AdS_3 metric from (2.17), we again require that that $L = \lambda e^{2r}$, with r the AdS radial coordinate and λ the AdS warp factor, which we require to be independent of the AdS coordinates. As before, we must now pick out the AdS radial direction $\hat{r} = \lambda^{-1/2}\mathrm{d}r$ from the space transverse to the Minkowski factor. In the generic case of interest to us, the AdS radial direction is a linear combination of the overall transverse direction $e^9 = C\mathrm{d}t$ and a radial direction in $\mathcal{M}_8$ transverse to the cycle that we denote by e^8. Thus, as before, we write the frame rotation relating

[5] In [41], somewhat more general wrapped brane metrics were considered than those of this discussion. However the discussion of this section is sufficiently general for the applications of interest in this paper.

the Minkowski and AdS frames as

$$\begin{aligned}\hat{r} &= \sin\theta e^8 + \cos\theta e^9, \\ \hat{\rho} &= \cos\theta e^8 - \sin\theta e^9,\end{aligned} \tag{2.18}$$

for a local rotation angle θ which we take to be independent of the AdS radial coordinate. Imposing AdS isometries on the electric and magnetic flux, and requiring that the metric on the space transverse to the AdS factor is independent of the AdS coordinates, we find that we may introduce an AdS frame coordinate ρ such that

$$\begin{aligned}\lambda^{-3/2}\cos\theta &= f(\rho), \\ \hat{\rho} &= \frac{\lambda}{2\sqrt{1-\lambda^3 f^2}}\mathrm{d}\rho,\end{aligned} \tag{2.19}$$

for some arbitrary function $f(\rho)$. See [41] for a fuller discussion of this point. Then the general AdS metric contained in (2.17) is

$$\mathrm{d}s^2 = \frac{1}{\lambda}\left[\mathrm{d}s^2(AdS_3) + \frac{\lambda^3}{4(1-\lambda^3 f^2)}\mathrm{d}\rho^2\right] + \mathrm{d}s^2(\mathcal{M}_7), \tag{2.20}$$

where $\mathrm{d}s^2(\mathcal{M}_7)$ is defined by

$$\mathrm{d}s^2(\mathcal{M}_8) = \mathrm{d}s^2(\mathcal{M}_7) + e^8 \otimes e^8. \tag{2.21}$$

The basis one-forms of the Minkowski frame are given in terms of the basis one-forms of the AdS frame by

$$\begin{aligned}e^8 &= \lambda\left(\sqrt{\frac{1-\lambda^3 f^2}{\lambda^3}}\mathrm{d}r + \sqrt{\frac{\lambda^3}{1-\lambda^3 f^2}}\frac{f}{2}\mathrm{d}\rho\right), \\ C\mathrm{d}t &= \lambda f \mathrm{d}r - \frac{1}{2}\lambda\mathrm{d}\rho.\end{aligned} \tag{2.22}$$

For an explicit AdS_3 solution we know λ and f explicitly, and so we can integrate these expressions to get an explicit coordinatisation of the AdS solution in the Minkowski frame. Thus we can freely pass between the canonical AdS and Minkowski frames for known AdS solutions.

As in the previous subsection, because we are picking out a preferred direction on $\mathcal{M}_8$, the Minkowski-frame structure on $\mathcal{M}_8$ is reduced, in the AdS frame, to a structure on $\mathcal{M}_7$. A $Spin(7)$ structure on $\mathcal{M}_8$ is reduced to a G_2 structure on $\mathcal{M}_7$; the decomposition of the Cayley four-form is

$$-\phi = \Upsilon + \Phi \wedge e^8. \tag{2.23}$$

An $SU(4)$ structure on $\mathcal{M}_8$ is reduced to an $SU(3)$ structure on $\mathcal{M}_7$. The decomposition of the $SU(4)$ structure forms is

$$\begin{aligned} J_8 &= J_6 + e^7 \wedge e^8, \\ \Omega_8 &= \Omega_6 \wedge (e^7 + ie^8), \end{aligned} \tag{2.24}$$

with

$$\mathrm{d}s^2(\mathcal{M}_8) = \mathrm{d}s^2(\mathcal{M}_7) + e^8 \otimes e^8 = \mathrm{d}s^2(\mathcal{M}_6) + e^7 \otimes e^7 + e^8 \otimes e^8, \tag{2.25}$$

with the $SU(3)$ structure forms defined on $\mathrm{d}s^2(\mathcal{M}_6)$. Finally, an $Sp(2)$ structure on $\mathcal{M}_8$ reduces to an $SU(2)$ structure on $\mathrm{d}s^2(\mathcal{M}_7)$. The decomposition of the triplet of $Sp(2)$ almost complex structures (which obey the algebra $J^A J^B = -\delta^{AB} + \epsilon^{ABC} J^C$, $A = 1, 2, 3$) under $SU(2)$ is

$$\begin{aligned} J^1 &= K^3 + e^5 \wedge e^6 + e^7 \wedge e^8, \\ J^2 &= K^2 - e^5 \wedge e^7 + e^6 \wedge e^8, \\ J^3 &= K^1 + e^6 \wedge e^7 + e^5 \wedge e^8, \end{aligned} \tag{2.26}$$

with

$$\mathrm{d}s^2(\mathcal{M}_8) = \mathrm{d}s^2(\mathcal{M}_4) + e^5 \otimes e^5 + e^6 \otimes e^6 + e^7 \otimes e^7 + e^8 \otimes e^8, \tag{2.27}$$

and the K^A are a triplet of self-dual $SU(2)$-invariant two-forms on $\mathcal{M}_4$, which satisfy the algebra[6] $K^A K^B = -\delta^{AB} - \epsilon^{ABC} K^C$. Having concluded the introductory review, we now move on to the main results of the paper.

3 Calabi-Yau interpolating pairs

In this section, we will give conjectured interpolating pairs for fivebranes wrapped on calibrated cycles in Calabi-Yau manifolds. First we will discuss Kähler cycles, then SLAG cycles. In order to present a complete picture, we will summarise the results of [31] for Kähler two-cycles in two-folds and three-folds. In the new cases, we will first present the pair, and then give the derivation of the special holonomy interpolation from the AdS solution.

3.1 Kähler cycles

In this subsection, the AdS solutions for which we give a conjectured special holonomy interpolation are: the half-BPS AdS_5 solution of [21], describing the near-horizon limit of fivebranes on

[6] The slightly eccentric labelling of the $SU(2)$ structure forms is chosen to coincide with an unfortunate conventional quirk of [41].

a two-cycle in a two-fold; the quarter-BPS AdS_5 solution of [21], for a two-cycle in a three-fold; and the AdS_3 solution of [24], admitting four Killing spinors, for a four-cycle in a four-fold. The special holonomy interpolations of the first two cases are derived in [31]; here we will just describe the conjectured pair. All the other pairs given in this paper are new, and their derivation will be given.

3.1.1 Two-fold

The conjectured interpolating pair The metric of the half-BPS AdS_5 solution of [21] is given by

$$
\begin{aligned}
\mathrm{d}s^2 &= \frac{1}{\lambda}\left[\mathrm{d}s^2(AdS_5) + \frac{1}{2}\mathrm{d}s^2(H^2) + (1-\lambda^3\rho^2)(\mathrm{d}\psi - P)^2 + \frac{\lambda^3}{4}\left(\frac{\mathrm{d}\rho^2}{1-\lambda^3\rho^2} + \rho^2\mathrm{d}s^2(S^2)\right)\right], \\
\lambda^3 &= \frac{8}{1+4\rho^2},
\end{aligned}
\tag{3.1}
$$

where $\mathrm{d}P = \mathrm{Vol}[H^2]$. The range of the coordinate ρ, which without loss of generality we take to be non-negative, is $\rho \in [0, 1/2]$. At $\rho = 0$, the R-symmetry S^2 degenerates smoothly[7]. At $\rho = 1/2$, the R-symmetry $U(1)$, with coordinate ψ, degenerates smoothly, provided that ψ is identified with period 2π.

As discussed in the introduction, the conjectured special holonomy interpolation of this manifold is

$$\mathrm{d}s^2(\mathcal{N}_\tau) = \mathrm{d}s^2(\mathbb{R}^{1,6}) + \mathrm{d}s^2(\mathcal{N}_\tau), \tag{3.2}$$

where, up to an overall scale,

$$\mathrm{d}s^2(\mathcal{N}_\tau) = \frac{R^2}{4}\left[\mathrm{d}s^2(H^2) + \left(\frac{1}{R^4} - 1\right)(\mathrm{d}\psi - P)^2\right] + \left(\frac{1}{R^4} - 1\right)^{-1}\mathrm{d}R^2. \tag{3.3}$$

The range of R is $R \in (0, 1]$. At $R = 1$, an S^2 degenerates smoothly, provided that ψ has the same period as in the AdS solution. At $R = 0$, the metric is singular, where the Kähler H^2 cycle degenerates.

[7] The R-symmetry of the dual theory is $SU(2) \times U(1)$.

3.1.2 Three-fold

The conjectured interpolating pair The metric of the quarter-BPS AdS_5 solution of [21] is

$$\mathrm{d}s^2 = \frac{1}{\lambda}\left[\mathrm{d}s^2(AdS_5) + \frac{1}{3}\mathrm{d}s^2(H^2) + \frac{1}{9}(1-\lambda^3\rho^2)\Big(\mathrm{d}s^2(S^2) + (\mathrm{d}\psi + P - P')^2\Big) + \frac{\lambda^3}{4(1-\lambda^3\rho^2)}\mathrm{d}\rho^2\right],$$

$$\lambda = \frac{4}{4+\rho^2}, \tag{3.4}$$

where now $\mathrm{d}P = \mathrm{Vol}[S^2]$, $\mathrm{d}P' = \mathrm{Vol}[H^2]$. This time, the range of ρ is $[-2/\sqrt{3}, 2/\sqrt{3}]$; at $\rho = \pm 2/\sqrt{3}$, an S^3 degenerates smoothly, provided that ψ is periodically identified with period 4π.

The conjectured special holonomy interpolation of this manifold is

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathbb{R}^{1,4}) + \mathrm{d}s^2(\mathcal{N}_\tau), \tag{3.5}$$

where, up to an overall scale,

$$\mathrm{d}s^2(\mathcal{N}_\tau) = \frac{1}{2}(1+\sin\xi)\mathrm{d}s^2(H^2) + \frac{\cos^2\xi}{2(1+\sin\xi)}\mathrm{d}s^2(S^2) + \frac{1}{\cos^2\xi}\Big(\mathrm{d}R^2 + R^2(\mathrm{d}\psi + P - P')^2\Big),$$

$$-\frac{1}{3}\sin^3\xi + \sin\xi = \frac{2}{3} - R^2. \tag{3.6}$$

The range of R is $R \in [0, 2/\sqrt{3})$. At $R = 0$ (corresponding to $\xi = \pi/2$) an S^3 degenerates smoothly, provided that ψ has the same periodicity as for the AdS coordinate. The metric is singular at $R = 2/\sqrt{3}$ (corresponding to $\xi = -\pi/2$) where the Kähler H^2 cycle degenerates. This metric is the hyperbolic analogue of the resolved conifold metric of [45], [46].

3.1.3 Four-folds

The interpolating pairs This is the first new case we encounter. A set of AdS_3 solutions was constructed by Gauntlett, Kim and Waldram (GKW) in [24], that describe the near-horizon limit of M5 branes on a Kähler four-cycle in a Calabi-Yau four-fold, intersecting membranes extended in the directions transverse to the four-fold. The AdS solutions admit four Killing spinors, and are as follows. The metrics are

$$\mathrm{d}s^2 = \frac{1}{\lambda}\left[\mathrm{d}s^2(AdS_3) + \frac{3}{4}\mathrm{d}s^2(\mathrm{KE}_4^-) + \frac{1}{4}(1-\lambda^3 f^2)\Big(\mathrm{d}s^2(S^2) + (\mathrm{d}\psi + P + P')^2\Big) + \frac{\lambda^3}{4(1-\lambda^3 f^2)}\mathrm{d}\rho^2\right],$$

$$\lambda^3 = \frac{9}{12+\rho^2}, \quad f = \frac{2\rho}{3}. \tag{3.7}$$

Here KE_4^- is an arbitrary negative scalar curvature Kähler-Einstein manifold, normalised such that the Ricci form $\mathcal{R}_4$ is given by $\mathcal{R}_4 = -\hat{J}_4$, with $\hat{J}_4$ the Kähler form of KE_4^-. In addition,

$$\begin{aligned} \mathrm{d}P &= \mathrm{Vol}[S^2], \\ \mathrm{d}P' &= \mathcal{R}_4. \end{aligned} \tag{3.8}$$

The range of ρ is $\rho \in [-2, 2]$; at the end-points, an S^3 smoothly degenerates, provided that ψ is periodically identified with period 4π. These manifolds admit an $SU(3)$ structure, which was obtained in [41], and will be given below (in somewhat more transparent coordinates), together with the magnetic flux (the electric flux, which is irrelevant to the discussion, can be obtained from [24] or [41]).

The conjectured special holonomy interpolation of these manifolds is

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathbb{R}^{1,2}) + \mathrm{d}s^2(\mathcal{N}_\tau), \tag{3.9}$$

where, up to an overall scale,

$$\begin{aligned} \mathrm{d}s^2(\mathcal{N}_\tau) &= \frac{1}{2}(1+\sin\xi)\mathrm{d}s^2(\mathrm{KE}_4^-) + \frac{\cos^2\xi}{2(1+\sin\xi)}\mathrm{d}s^2(S^2) + \frac{1}{\cos^2\xi}\Big(\mathrm{d}R^2 + R^2(\mathrm{d}\psi + P - P')^2\Big), \\ -\frac{1}{3}\sin^3\xi + \sin\xi &= \frac{2}{3} - R^2. \end{aligned} \tag{3.10}$$

This is identical to the three-fold metric of the previous subsection, but with the H^2 replaced by a KE_4^-. It has the same regularity properties, and is the hyperbolic analogue of the four-fold resolved conifold. Now we will discuss its derivation.

The G-structure of the AdS solutions First we will give the $SU(3)$ structure of the AdS solutions, defined by all four Killing spinors. Defining the frame

$$\begin{aligned} e^a &= \sqrt{\frac{3}{4\lambda}}\hat{e}^a, \\ e^5 + ie^6 &= \frac{1}{2}\sqrt{1-\lambda^3 f^2}e^{i\psi}(\mathrm{d}\theta + i\sin\theta\mathrm{d}\phi), \\ e^7 &= \frac{1}{2}\sqrt{1-\lambda^3 f^2}(\mathrm{d}\psi + P + P'), \end{aligned} \tag{3.11}$$

where $a = 1, ..., 4$, the $\hat{e}^a$ furnish a basis for KE_4^-, $\hat{J}_4 = \hat{e}^{12} + \hat{e}^{34}$ and $\hat{\Omega}_4 = (\hat{e}^1 + i\hat{e^2})(\hat{e}^3 + i\hat{e}^4)$, the $SU(3)$ structure is given by

$$\begin{aligned} J_6 &= e^{12} + e^{34} + e^{56}, \\ \Omega_6 &= (e^1 + ie^2)(e^3 + ie^4)(e^5 + ie^6). \end{aligned} \tag{3.12}$$

This structure is a solution of the torsion conditions of [41] for the near-horizon limit of fivebranes on a Kähler four-cycle in a four-fold, which are

$$\hat{\rho}\wedge \mathrm{d}(\lambda^{-1}J_6\wedge J_6) \;=\; 0, \tag{3.13}$$

$$\mathrm{d}(\lambda^{-3/2}\sqrt{1-\lambda^3 f^2}\mathrm{Im}\Omega_6) \;=\; 2\lambda^{-1}(e^7\wedge \mathrm{Re}\Omega_6 - \lambda^{3/2}f\hat{\rho}\wedge \mathrm{Im}\Omega_6), \tag{3.14}$$

$$J_6 \lrcorner\, \mathrm{d}e^7 \;=\; \frac{2\lambda^{1/2}}{\sqrt{1-\lambda^3 f^2}}(1-\lambda^3 f^2) - \lambda^{3/2}f\hat{\rho}\lrcorner\, \mathrm{d}\log\left(\frac{\lambda^3 f}{1-\lambda^3 f^2}\right). \tag{3.15}$$

In addition it is a solution of the Bianchi identity for the magnetic flux, $\mathrm{d}F_{\mathrm{mag}}=0$, which in this case is not implied by the torsion conditions. The magnetic flux is given by

$$F_{\mathrm{mag}} \;=\; \frac{\lambda^{3/2}}{\sqrt{1-\lambda^3 f^2}}(\lambda^{3/2}f+\star_8)(\mathrm{d}[\lambda^{-3/2}\sqrt{1-\lambda^3 f^2}J_6\wedge e^7] - 2\lambda^{-1}J_6\wedge J_6) + 2\lambda^{1/2}J_6\wedge e^7\wedge\hat{\rho}, \tag{3.16}$$

where $\star_8$ is the Hodge dual on the space transverse to the AdS factor, with positive orientation defined with repect to

$$\mathrm{Vol} = \frac{1}{3!}J_6\wedge J_6\wedge J_6\wedge e^7\wedge\hat{\rho}. \tag{3.17}$$

The AdS solutions in the Minkowski frame Now we use the discussion of section 2 to frame-rotate the AdS solutions to the canonical Minkowski frame. Defining the coordinates

$$\begin{aligned} t &= -\frac{1}{2}e^{-4r/3}\rho, \\ u &= -\frac{1}{3}\sqrt{12-3\rho^2}e^{-r}, \end{aligned} \tag{3.18}$$

the one-forms e^8, e^9 in the Minkowski frame are given by

$$\begin{aligned} e^8 &= \lambda e^r \mathrm{d}u, \\ e^9 &= \lambda e^{4r/3}\mathrm{d}t, \end{aligned} \tag{3.19}$$

and the metric in the Minkowski frame takes the form

$$\begin{aligned} \mathrm{d}s^2 \;=\; & \frac{1}{H_{M5}^{1/3}H_{M2}^{2/3}}\mathrm{d}s^2(\mathbb{R}^{1,1}) + \frac{H_{M5}^{2/3}}{H_{M2}^{2/3}}\mathrm{d}t^2 + \frac{H_{M2}^{1/3}}{H_{M5}^{1/3}}\left[\frac{3}{4}F\mathrm{d}s^2(\mathrm{KE}_4^-)\right] \\ & + H_{M2}^{1/3}H_{M5}^{2/3}\left[\frac{1}{F}\left(\mathrm{d}u^2 + \frac{u^2}{4}[\mathrm{d}s^2(S^2) + (\mathrm{d}\psi + P + P')^2]\right)\right], \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} H_{M5} &= \lambda^3 e^{14r/3}, \\ H_{M2} &= e^{2r/3}, \\ F &= e^{4r/3}. \end{aligned} \tag{3.21}$$

These three functions have been chosen so that the metric takes a form reminiscent of the harmonic function superposition rule for intersecting branes, in line with the probe brane picture. The fivebrane worldvolume directions are the Minkowski and KE_4^- directions; the membranes extend along the Minkowski and t directions. Also e^{2r} is given in terms of t and u by a positive signature metric inducing root of the quartic

$$t^6 e^{8r} - \left(1 - \frac{3}{4}u^2 e^{2r}\right)^3 = 0. \tag{3.22}$$

The wrapped-brane $SU(4)$ structure of the AdS_3 solutions, defined by two of their Killing spinors, is given by

$$\begin{aligned} J_8 &= J_6 + e^7 \wedge e^8, \\ \Omega_8 &= \Omega_6 \wedge (e^7 + ie^8). \end{aligned} \tag{3.23}$$

By construction, this structure is a solution of the wrapped brane equations for a Kähler four-cycle in a four-fold. These comprise the torsion conditions [60], [41]

$$\begin{aligned} J_8 \lrcorner\, \mathrm{d}e^9 &= 0, \\ \mathrm{d}(L^{-1}\mathrm{Re}\Omega_8) &= 0, \\ e^9 \wedge [J_8 \lrcorner\, \mathrm{d}J_8 - Le^9 \lrcorner\, \mathrm{d}(L^{-1}e^9)] &= 0, \end{aligned} \tag{3.24}$$

and the Bianchi identity and field equation for the four-form, which is given in the Minkowski frame in [60], [41].

The conjectured Calabi-Yau interpolation We now make the following ansatz for an interpolating solution:

$$\begin{aligned} \mathrm{d}s^2 = &\ \frac{1}{H_{M5}^{1/3}H_{M2}^{2/3}}\mathrm{d}s^2(\mathbb{R}^{1,1}) + \frac{H_{M5}^{2/3}}{H_{M2}^{2/3}}\mathrm{d}t^2 + \frac{H_{M2}^{1/3}}{H_{M5}^{1/3}}\left[\alpha^2 F_1^2 F_2^2 \mathrm{d}s^2(\mathrm{KE}_4^-)\right] \\ &+ H_{M2}^{1/3}H_{M5}^{2/3}\left[\frac{1}{F_1^2}\left(\mathrm{d}u^2 + \frac{u^2}{4}(\mathrm{d}\psi + P + P')^2\right) + \frac{u^2}{4F_2^2}\mathrm{d}s^2(S^2)\right], \end{aligned} \tag{3.25}$$

with $H_{M5,M2}$, $F_{1,2}$ arbitrary functions of u, t, and α a constant. To determine the Calabi-Yau interpolation with this ansatz, we set $H_{M5,M2} = 1$ and require that $F_{1,2}$ are functions only of u. The derivation of the Calabi-Yau metric is now identical to that for the three-fold interpolation of the previous subsection, as given in [31]. This close analogy between fivebranes wrapped on Kähler four-cycles in four-folds and two-cycles in three-folds has recently been used to construct infinite families of AdS_3 solutions [28], [29], [30] motivated by the analogous AdS_5 solutions [13].

In any event, to determine the special holonomy metric, observe that closure of Ω_8, with the obvious frame inherited from the AdS solution, is automatic. Closure of J_8 results in the pair of equations

$$\begin{aligned}
\alpha^2\partial_u(F_1^2F_2^2) + \frac{u}{2F_1^2} &= 0, \\
\partial_u\left(\frac{u^2}{4F_2^2}\right) - \frac{u}{2F_1^2} &= 0.
\end{aligned} \tag{3.26}$$

As in [31], [59], the general solution of these equations inducing a metric with only one singular degeneration point is given by

$$\begin{aligned}
F_1^2 &= \frac{a^4}{\alpha^2u^2}\cos^2\xi, \\
F_2^2 &= \frac{u^2}{2a^2}\frac{(1+\sin\xi)}{\cos^2\xi}, \\
-\frac{1}{3}\sin^2\xi + \sin\xi &= \frac{2}{3} - \frac{\alpha^2u^4}{4a^6},
\end{aligned} \tag{3.27}$$

for some constant α. Defining the coordinate

$$R^2 = \frac{\alpha^2u^4}{4a^6}, \tag{3.28}$$

the metric takes the form given above.

3.2 Special Lagrangian Cycles

In this subsection we will give conjectured interpolating pairs for fivebranes wrapped on SLAG cycles in three- and four-folds. The AdS solutions for which a Calabi-Yau interpolation is derived are respectively the AdS_4 solution of [23], admitting eight Killing spinors; and the AdS_3 solution of [24], admitting four Killing spinors. In each case we will first give the conjectured pair, then the derivation of the Calabi-Yau interpolation from the AdS solution.

3.2.1 Three-fold

The interpolating pair The eleven-dimensional lift of the AdS_4 solution of [23] was later interpreted [24] as the near-horizon limit of fivebranes wrapped on a SLAG three-cycle in a three-fold. The metric is given by

$$\mathrm{d}s^2 = \frac{1}{\lambda}\left[\mathrm{d}s^2(AdS_4) + \mathrm{d}s^2(H^3) + (1-\lambda^3\rho^2)\mathrm{D}Y^a\mathrm{D}Y^a + \frac{\lambda^3}{4(1-\lambda^3\rho^2)}\left(\mathrm{d}\rho^2 + \rho^2\mathrm{d}s^2(S^1)\right)\right],$$

$$\lambda^3 = \frac{2}{8+\rho^2}. \tag{3.29}$$

The flux, which in this case is purely magnetic and irrelevant to the discussion, may be obtained from [24] or [39]. Here the Y^a, $a = 1, 2, 3$, are constrained coordinates on an S^2, $Y^aY^a = 1$, and

$$\mathrm{D}Y^a = \mathrm{d}Y^a + \omega^a{}_b Y^b, \tag{3.30}$$

where the ω_{ab} are the spin-connection one-forms of H^3. The range of ρ, which without loss of generality we take to be positive, is $\rho \in [0, \sqrt{8}]$. At $\rho = 0$ the R-symmetry S^1 degenerates smoothly[8], while at $\rho = \sqrt{8}$ the S^2 degenerates smoothly.

Denoting a basis for H^3 by e^a, the metric of the conjectured Calabi-Yau interpolation of this solution is

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathbb{R}^{1,4}) + \mathrm{d}s^2(\mathcal{N}_\tau), \tag{3.31}$$

where, up to an overall scale,

$$\begin{aligned}\mathrm{d}s^2(\mathcal{N}_\tau) \quad = \quad & \frac{(2\theta - \sin 2\theta)^{1/3}}{\sin\theta}\Big[\frac{1}{2}(1-\cos\theta)(e^a - Y^aY^be^b)^2 + \frac{1}{2}(1+\cos\theta)\mathrm{D}Y^a\mathrm{D}Y^a \\ & + \frac{1}{3}\Big(\frac{\sin^3\theta}{2\theta - \sin 2\theta}\Big)\Big(\mathrm{d}\theta^2 + 4(Y^ae^a)^2\Big)\Big]. \end{aligned} \tag{3.32}$$

The range of θ is $\theta \in (0, \pi]$. Near $\theta = \pi$, the S^2 degenerates smoothly; up to a scale, near $\theta = \pi$ the metric is

$$\mathrm{d}s^2 = \mathrm{d}s^2(H^3) + \frac{1}{4}\Big[\mathrm{d}\theta^2 + \theta^2\mathrm{D}Y^a\mathrm{D}Y^a\Big]. \tag{3.33}$$

The metric is singular at $\theta = 0$; up to a scale, near $\theta = 0$ it is

$$\mathrm{d}s^2 = \frac{1}{4}\Big[\mathrm{d}\theta^2 + \theta^2(e^a - Y^aY^be^b)^2\Big] + (Y^ae^a)^2 + \mathrm{D}Y^a\mathrm{D}Y^a. \tag{3.34}$$

This Calabi-Yau is the hyperbolic analogue of the deformed conifold [45] (which coincides with the Stenzel three-fold metric [48]); the S^3 SLAG cycle of the deformed conifold is replaced by a H^3 in the $\mathcal{N}_\tau$ metric. Now we discuss the derivation of this interpolation from the AdS solution.

[8]The R-symmetry of the dual conformal theory is $U(1)$.

The G-structure of the AdS solution The AdS_4 solution admits admits an $SU(2)$ structure defined by all eight Killing spinors. It is given by [39]

$$
\begin{aligned}
e^5 &= \frac{1}{\lambda^{1/2}} Y^a e^a, \\
J^1 &= \frac{1}{\lambda}\sqrt{1-\lambda^3\rho^2}\,\mathrm{D}Y^a \wedge e^a, \\
J^2 &= \frac{1}{\lambda}\sqrt{1-\lambda^3\rho^2}\,\epsilon^{abc} Y^a \mathrm{D}Y^b \wedge e^c, \\
J^3 &= \frac{1}{2}\epsilon^{abc}\left[\frac{1}{\lambda}(1-\lambda^3\rho^2) Y^a \mathrm{D}Y^b \wedge \mathrm{D}Y^c - \frac{1}{\lambda} Y^a e^b \wedge e^c\right].
\end{aligned} \tag{3.35}
$$

This structure satisfies the torsion conditions of [39] for the near-horizon limit of fivebranes on a SLAG three-cycle in a three-fold, which are

$$
\begin{aligned}
\mathrm{d}\left(\lambda^{-1}\sqrt{1-\lambda^3\rho^2}\,e^5\right) &= \lambda^{-1/2} J^1 + \lambda\rho e^5 \wedge \hat{\rho}, \\
\mathrm{d}\left(\lambda^{-3/2} J^3 \wedge e^5 - \rho J^2 \wedge \hat{\rho}\right) &= 0, \\
\mathrm{d}\left(J^2 \wedge e^5 + \lambda^{-3/2}\rho^{-1} J^3 \wedge \hat{\rho}\right) &= 0.
\end{aligned} \tag{3.36}
$$

The following identities, valid for a H^3 or S^3 with scalar curvature R, are useful in verifying this claim:

$$
\begin{aligned}
\mathrm{d}(Y^a e^a) &= \mathrm{D}Y^a \wedge e^a, \\
\mathrm{d}(\epsilon^{abc} Y^a \mathrm{D}Y^b \wedge \mathrm{D}Y^c) &= -\frac{R}{3}\epsilon^{abc} Y^a \mathrm{D}Y^b \wedge e^c \wedge Y^d e^d, \\
\mathrm{d}(\epsilon^{abc} Y^a e^b \wedge e^c) &= 2\epsilon^{abc} Y^a \mathrm{D}Y^b \wedge e^c \wedge Y^d e^d, \\
\mathrm{d}(\epsilon^{abc} Y^a \mathrm{D}Y^b \wedge e^c) &= \epsilon^{abc}\left[Y^a \mathrm{D}Y^b \wedge \mathrm{D}Y^c - \frac{R}{6} Y^a e^b \wedge e^c\right] \wedge Y^d e^d.
\end{aligned} \tag{3.37}
$$

In this case, the Bianchi identity for the flux is implied by the torsion conditions [39].

The AdS solution in the Minkowski frame From section 2, defining the Minkowski-frame coordinates

$$
\begin{aligned}
t &= -\frac{\rho}{2} e^{-2r}, \\
u &= -\sqrt{\frac{8-\rho^2}{2}}\, e^{-r},
\end{aligned} \tag{3.38}
$$

the metric of the AdS solution in the Minkowski frame is given by

$$
\mathrm{d}s^2 = L^{-1}\Big[\mathrm{d}s^2(\mathbb{R}^{1,2}) + F\mathrm{d}s^2(H^3)\Big] + L^2\Big[F^{-1}(\mathrm{d}u^2 + u^2 \mathrm{D}Y^a \mathrm{D}Y^a) + \mathrm{d}s^2(\mathbb{R}^2)\Big], \tag{3.39}
$$

where $L = \lambda e^{2r}$, $F = e^{2r}$ and

$$e^{2r} = \frac{u^2}{4t^2}\left(-1 + \sqrt{1 + 32t^2/u^2}\right). \tag{3.40}$$

The wrapped-brane $SU(3)$ structure of the AdS solution, defined by four of its Killing spinors, is given by

$$\begin{aligned} J_6 &= J^1 + e^5 \wedge \hat{u}, \\ \Omega_6 &= (J^3 + iJ^2) \wedge (e^5 + i\hat{u}), \end{aligned} \tag{3.41}$$

with $\hat{u} = LF^{-1/2}\mathrm{d}u$. By construction, this structure is a solution of the wrapped brane equations for fivebranes wrapped on a SLAG cycle in a three-fold, which are [37]

$$\begin{aligned} \mathrm{Vol}[\mathbb{R}^2] \wedge \mathrm{dIm}\Omega_6 &= 0, \\ \mathrm{d}(L^{-1/2}J_6) &= 0, \\ \mathrm{Re}\Omega_6 \wedge \mathrm{dRe}\Omega_6 &= 0, \\ \mathrm{d}\Big(\star_8 L^{3/2}\mathrm{d}(L^{-3/2}\mathrm{Re}\Omega_6)\Big) &= 0, \end{aligned} \tag{3.42}$$

where $\star_8$ denotes the Hodge dual on the space transverse to the Minkowski factor.

The conjectured Calabi-Yau interpolation We now make the following ansatz for an interpolating solution:

$$\mathrm{d}s^2 = L^{-1}\Big[\mathrm{d}s^2(\mathbb{R}^{1,2}) + F_1^2(e^a - Y^aY^be^b)^2 + F_2^2(Y^ae^a)^2\Big] + L^2\Big[F_4^2\mathrm{d}u^2 + F_3^2\mathrm{D}Y^a\mathrm{D}Y^a + \mathrm{d}s^2(\mathbb{R}^2)\Big], \tag{3.43}$$

with L, $F_{1,\dots,4}$ arbitrary functions of u and t. To determine the Calabi-Yau interpolation with this ansatz, we set $L = 1$, and require that $F_{1,\dots,4}$ are functions only of u. Then F_4 is at our disposal and we set it to unity. The Calabi-Yau condition is

$$\mathrm{d}J_6 = \mathrm{d}\Omega_6 = 0, \tag{3.44}$$

with J_6 and Ω_6 as inherited from the AdS solution in the Minkowski frame,

$$\begin{aligned} J_6 &= F_1F_3\mathrm{D}Y^a \wedge e^a + F_2Y^ae^a \wedge \mathrm{d}u, \\ \mathrm{Re}\Omega_6 &= \frac{1}{2}\Big(F_2F_3^2\epsilon^{abc}Y^a\mathrm{D}Y^b \wedge \mathrm{D}Y^c - F_1^2F_2\epsilon^{abc}Y^ae^b \wedge e^c\Big) \wedge Y^de^d - F_1F_3\epsilon^{abc}Y^a\mathrm{D}Y^b \wedge e^c, \\ \mathrm{Im}\Omega_6 &= F_1F_2F_3\epsilon^{abc}Y^a\mathrm{D}Y^b \wedge e^c \wedge Y^de^d + \frac{1}{2}\Big(F_3^2\epsilon^{abc}Y^a\mathrm{D}Y^b \wedge \mathrm{D}Y^c - F_1^2\epsilon^{abc}Y^ae^b \wedge e^c\Big) \wedge \mathrm{d}u. \end{aligned} \tag{3.45}$$

Then using the equations (3.37), closure of J_6 implies

$$\partial_u(F_1F_3) + F_2 = 0. \tag{3.46}$$

Closure of $\mathrm{Re}\Omega_6$ implies

$$\begin{aligned} \frac{1}{2}\partial_u(F_2F_3^2) + F_1F_3 &= 0, \\ \frac{1}{2}\partial_u(F_2F_1^2) - F_1F_3 &= 0. \end{aligned} \tag{3.47}$$

Closure of $\mathrm{Im}\Omega_6$ implies

$$\partial_u(F_1F_2F_3) - F_3^2 + F_1^2 = 0, \tag{3.48}$$

and this equation is implied by the other three. Solving (3.46) and (3.47) is straightforward. Adding (3.47) we immediately get

$$F_2 = \frac{a}{F_1^2 + F_3^2}, \tag{3.49}$$

for constant a. Next, subtracting (3.47), and defining a new coordinate x according to

$$\partial_u = -\frac{4}{a}F_1F_3\partial_x, \tag{3.50}$$

we get

$$\frac{F_3^2 - F_1^2}{F_1^2 + F_3^2} = x + b, \tag{3.51}$$

for a constant b which may be eliminated by a shift of x. Solving for F_3, inserting in (3.46), and defining $x = \cos\theta$, we obtain

$$F_1^6 = \frac{3a^2}{32}(2\theta - \sin 2\theta + c)\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{3/2}, \tag{3.52}$$

for constant c. The metric has pathological behaviour unless $c = 0$, so we choose this value. Then, up to an overall scale of $(3a^2/4)^{1/3}$, we obtain the three-fold metric given above.

3.2.2 Four-folds

The interpolating pair The GKW solution for AdS_3 near-horizon limit of a string intersection of fivebranes wrapped on a SLAG four-cycle in a four-fold, with membranes extended in the directions transverse to the four-fold, was constructed in [24]. The metric is given by

$$ds^2 = \frac{1}{\lambda}\left[ds^2(AdS_3) + \frac{8}{3}ds^2(H^4) + (1-\lambda^3 f^2)DY^aDY^a + \frac{\lambda^3}{4(1-\lambda^3 f^2)}d\rho^2\right],$$

$$\lambda^3 = \frac{16}{24+3\rho^2}, \quad f = \frac{3\rho}{4}. \tag{3.53}$$

Here the Y^a, $a = 1, ..., 4$ are constrained coordinates on a three-sphere, $Y^aY^a = 1$, and

$$\mathrm{D}Y^a = \mathrm{d}Y^a + \omega^a{}_b Y^b, \tag{3.54}$$

with ω_{ab} the spin connection one-forms of H^4. The range of ρ is $\rho \in [-2, 2]$; at the endpoints, the S^3 degenerates smoothly. The electric flux may be obtained from [24] or [41]; the magnetic flux will be given below.

Denoting a basis for H^4 by e^a, the metric of the conjectured Calabi-Yau interpolation of this solution is

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathbb{R}^{1,2}) + \mathrm{d}s^2(\mathcal{N}_\tau), \tag{3.55}$$

where, up to an overall scale,

$$\begin{aligned} \mathrm{d}s^2(\mathcal{N}_\tau) \quad = \quad & \frac{(2+\cos 2\theta)^{1/4}}{\cos\theta}\Big[\cos^2\theta(e^a - Y^aY^be^b)^2 + \sin^2\theta \mathrm{D}Y^a\mathrm{D}Y^a \\ & + \frac{3\cos\theta\sin^3 2\theta}{8\sin^3\theta(2+\cos 2\theta)}\Big(\mathrm{d}\theta^2 + (Y^ae^a)^2\Big)\Big]. \end{aligned} \tag{3.56}$$

This metric is the hyperbolic analogue of the Stenzel four-fold. Without loss of generality, we can take the range of θ to be $\theta \in [0, \pi/2)$. Near $\theta = 0$, the S^3 degenerates smoothly, and up to a scale the metric is given by

$$\mathrm{d}s^2 = \mathrm{d}s^2(H^4) + \mathrm{d}\theta^2 + \theta^2 \mathrm{D}Y^a\mathrm{D}Y^a. \tag{3.57}$$

The other degeneration point, $\theta = \pi/2$, is singular. Now we give the derivation of the conjectured interpolation.

The G-structure of the AdS solution The AdS_3 solution admits an $SU(3)$ structure defined by all four Killing spinors. The structure satisfies the torsion conditions of [41] for the near-horizon limit of fivebranes on a SLAG four-cycle in a four-fold, together with the Bianchi identity for the magnetic flux, $\mathrm{d}F_{\mathrm{mag}} = 0$, which in this case is not implied by the torsion conditions. The $SU(3)$

structure is [41]

$$
\begin{aligned}
e^7 &= -\sqrt{\frac{8}{3\lambda}} Y^a e^a, \\
J_6 &= \sqrt{\frac{8(1-\lambda^3 f^2)}{3\lambda^2}} e^a \wedge \mathrm{D}Y^a, \\
\mathrm{Re}\Omega_6 &= \left(\sqrt{\frac{8}{3\lambda}}\right)^3 \frac{1}{3!}\epsilon^{abcd} Y^a e^b \wedge e^c \wedge e^d - \sqrt{\frac{8}{3\lambda^3}}\left(1-\lambda^3 f^2\right)\frac{1}{2}\epsilon^{abcd} Y^a \mathrm{D}Y^b \wedge \mathrm{D}Y^c \wedge e^d, \\
\mathrm{Im}\Omega_6 &= \frac{8}{3}\sqrt{\frac{1-\lambda^3 f^2}{\lambda^3}}\frac{1}{2}\epsilon^{abcd} Y^a \mathrm{D}Y^b \wedge e^c \wedge e^d - \left(\sqrt{\frac{1-\lambda^3 f^2}{\lambda}}\right)^3 \frac{1}{3!}\epsilon^{abcd} Y^a \mathrm{D}Y^b \wedge \mathrm{D}Y^c \wedge \mathrm{D}Y^d.
\end{aligned}
\tag{3.58}
$$

The torsion conditions are

$$
e^7 \wedge \hat{\rho} \wedge \mathrm{d}\left(\frac{\mathrm{Re}\Omega_6}{\sqrt{1-\lambda^3 f^2}}\right) = 0, \tag{3.59}
$$

$$
\mathrm{d}\left(\lambda^{-1}\sqrt{1-\lambda^3 f^2} e^7\right) = \lambda^{-1/2}(J_6 + \lambda^{3/2} f e^7 \wedge \hat{\rho}), \tag{3.60}
$$

$$
\mathrm{Im}\Omega_6 \wedge \mathrm{d}\mathrm{Im}\Omega_6 = \frac{\lambda^{1/2}}{\sqrt{1-\lambda^3 f^2}}(6+4\lambda^3 f^2)\mathrm{Vol}[\mathcal{M}_6] \wedge e^7 - 2\lambda^{3/2} f \star_8 \mathrm{d}\log\left(\frac{\lambda^3 f}{1-\lambda^3 f^2}\right), \tag{3.61}
$$

where

$$
\mathrm{Vol}[\mathcal{M}_6] = \frac{1}{3!} J_6 \wedge J_6 \wedge J_6. \tag{3.62}
$$

and $\star_8$ denotes the Hodge dual on the space transverse to the AdS factor, with positive orientation defined with respect to

$$
\mathrm{Vol} = \mathrm{Vol}[\mathcal{M}_6] \wedge e^7 \wedge \hat{\rho}. \tag{3.63}
$$

The magnetic flux is

$$
F_{\mathrm{mag}} = -\frac{\lambda^{3/2}}{1-\lambda^3 f^2}(\lambda^{3/2} f + \star_8)\left(\mathrm{d}\left[\lambda^{-3/2}\sqrt{1-\lambda^3 f^2}\mathrm{Im}\Omega_6\right] + 4\lambda^{-1}\mathrm{Re}\Omega_6 \wedge e^7\right) - 2\lambda^{1/2}\mathrm{Im}\Omega_6 \wedge \hat{\rho}. \tag{3.64}
$$

The following identities, valid for a H^4 or an S^4 with scalar curvature R, are useful in verifying the torsion conditions and Bianchi identity:

$$\begin{aligned}
\mathrm{d}\Big(\epsilon^{abcd}Y^a e^b\wedge e^c\wedge e^d\Big) &= -3\epsilon^{abcd}Y^a\mathrm{D}Y^b\wedge e^c\wedge e^d\wedge Y^e e^e,\\
\mathrm{d}\Big(\epsilon^{abcd}Y^a\mathrm{D}Y^b\wedge e^c\wedge e^d\Big) &= \Big(-2\epsilon^{abcd}Y^a\mathrm{D}Y^b\wedge\mathrm{D}Y^c\wedge e^d+\frac{R}{12}\epsilon^{abcd}Y^a e^b\wedge e^c\wedge e^d\Big)\wedge Y^e e^e,\\
\mathrm{d}\Big(\epsilon^{abcd}Y^a\mathrm{D}Y^b\wedge\mathrm{D}Y^c\wedge e^d\Big) &= \Big(\frac{R}{6}\epsilon^{abcd}Y^a\mathrm{D}Y^b\wedge e^c\wedge e^d-\epsilon^{abcd}Y^a\mathrm{D}Y^b\wedge\mathrm{D}Y^c\wedge\mathrm{D}Y^d\Big)\wedge Y^e e^e,\\
\mathrm{d}\Big(\epsilon^{abcd}Y^a\mathrm{D}Y^b\wedge\mathrm{D}Y^c\wedge\mathrm{D}Y^d\Big) &= \frac{R}{4}\epsilon^{abcd}Y^a\mathrm{D}Y^b\wedge\mathrm{D}Y^c\wedge e^d\wedge Y^e e^e.
\end{aligned}\tag{3.65}$$

The AdS solutions in the Minkowski frame Using section 2, we define the coordinates

$$\begin{aligned}
t &= -\frac{1}{2}e^{-3r/2}\rho,\\
u &= -\sqrt{\frac{24-6\rho^2}{16}}e^{-r},
\end{aligned}\tag{3.66}$$

so that the one-forms e^8, e^9 in the Minkowski frame are given by

$$\begin{aligned}
e^8 &= \lambda e^r\mathrm{d}u,\\
e^9 &= \lambda e^{3r/2}\mathrm{d}t,
\end{aligned}\tag{3.67}$$

and the AdS metric in the Minkowski frame takes the form

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{1}{H_{M5}^{1/3}H_{M2}^{2/3}}\mathrm{d}s^2(\mathbb{R}^{1,1})+\frac{H_{M5}^{2/3}}{H_{M2}^{2/3}}\mathrm{d}t^2+\frac{H_{M2}^{1/3}}{H_{M5}^{1/3}}\left[\frac{8}{3}F\mathrm{d}s^2(H^4)\right]\\
&\quad+H_{M2}^{1/3}H_{M5}^{2/3}\left[\frac{1}{F}\left(\mathrm{d}u^2+u^2\mathrm{D}Y^a\mathrm{D}Y^a\right)\right],
\end{aligned}\tag{3.68}$$

where

$$\begin{aligned}
H_{M5} &= \lambda^3 e^{5r},\\
H_{M2} &= e^{r/2},\\
F &= e^{3r/2}.
\end{aligned}\tag{3.69}$$

The function e^r is given in terms of t and u by a positive signature metric inducing root of the cubic

$$t^2e^{3r}+\frac{2}{3}u^2e^{2r}-1=0.\tag{3.70}$$

The wrapped brane $SU(4)$ structure of the AdS_3 solution, defined by two of its Killing spinors, is given by

$$\begin{aligned}
J_8 &= J_6+e^7\wedge e^8,\\
\Omega_8 &= \Omega_6\wedge(e^7+ie^8).
\end{aligned}\tag{3.71}$$

By construction, this structure is a solution of the wrapped brane equations for a SLAG four-cycle in a four-fold, which comprise the torsion conditions [41]

$$\begin{aligned} \mathrm{d}(L^{-1/2}J_8) &= 0, \\ \mathrm{Im}\Omega_8 \wedge \mathrm{dRe}\Omega_8 &= 0, \\ e^9 \wedge [\mathrm{Re}\Omega_8 \lrcorner\, \mathrm{dRe}\Omega_8 - 2L^{3/2}e^9 \lrcorner\, \mathrm{d}(L^{-3/2}e^9)] &= 0. \end{aligned} \tag{3.72}$$

together with the Bianchi identity and field equation for the four-form, which is given in [41].

The conjectured Calabi-Yau interpolation We make the following ansatz for an interpolating solution:

$$\begin{aligned} \mathrm{d}s^2 &= \frac{1}{H_{M5}^{1/3}H_{M2}^{2/3}}\mathrm{d}s^2(\mathbb{R}^{1,1}) + \frac{H_{M5}^{2/3}}{H_{M2}^{2/3}}\mathrm{d}t^2 + \frac{H_{M2}^{1/3}}{H_{M5}^{1/3}}\left[F_1^2(e^a - Y^aY^be^b)^2 + F_2^2(Y^be^b)^2\right] \\ &\quad + H_{M2}^{1/3}H_{M5}^{2/3}\left[F_4^2\mathrm{d}u^2 + F_3^2\mathrm{D}Y^a\mathrm{D}Y^a\right], \end{aligned} \tag{3.73}$$

with $H_{M5,M2}$, $F_{1,\ldots,4}$ arbitrary functions of u, t. To determine the Calabi-Yau interpolation with this ansatz, we set $H_{M5,M2} = 1$ and require that $F_{1,\ldots,4}$ are functions only of u. Then F_4 is at our disposal, and we set it to 1. Requiring $SU(4)$ holonomy, we set

$$\mathrm{d}J_8 = \mathrm{d}\Omega_8 = 0, \tag{3.74}$$

with

$$\begin{aligned} J_8 &= J_6 + e^7 \wedge \mathrm{d}u, \\ \Omega_8 &= \Omega_6 \wedge (e^7 + i\mathrm{d}u), \\ e^7 &= -F_2Y^ae^a, \\ J_6 &= F_1F_3e^a \wedge \mathrm{D}Y^a, \\ \mathrm{Re}\Omega_6 &= F_1^3\frac{1}{3!}\epsilon^{abcd}Y^ae^b \wedge e^c \wedge e^d - F_1F_3^2\frac{1}{2}\epsilon^{abcd}Y^a\mathrm{D}Y^b \wedge \mathrm{D}Y^c \wedge e^d, \\ \mathrm{Im}\Omega_6 &= F_1^2F_3\frac{1}{2}\epsilon^{abcd}Y^a\mathrm{D}Y^b \wedge e^c \wedge e^d - F_3^3\frac{1}{3!}\epsilon^{abcd}Y^a\mathrm{D}Y^b \wedge \mathrm{D}Y^c \wedge \mathrm{D}Y^d. \end{aligned} \tag{3.75}$$

Closure of J_8 implies

$$\partial_u(F_1F_3) + F_2 = 0. \tag{3.76}$$

Using the identities (3.65) with $R = -12$, closure of $\mathrm{Re}\Omega_8$ implies

$$\begin{aligned} \partial_u(F_1^3F_2) - 3F_1^2F_3 &= 0, \\ \partial_u(F_3^2F_2) + 3F_1F_3^2 &= 0, \end{aligned} \tag{3.77}$$

while closure of $\mathrm{Im}\Omega_8$ implies

$$\begin{aligned}\partial_u(F_1^2F_2F_3)+F_1^3-2F_1F_3^2 &= 0,\\ \partial_u(F_1F_2F_3^2)-F_3^3+2F_1^2F_3 &= 0.\end{aligned} \tag{3.78}$$

It may be verified that the last two equations are implied by the first three. Solving for $F_{1,2,3}$ is straightforward. First define a new coordinate x according to

$$-F_2\partial_u=\partial_\theta. \tag{3.79}$$

Then we have that

$$\partial_\theta\left(\frac{F_1}{F_3}\right)=-1-\left(\frac{F_1}{F_3}\right)^2, \tag{3.80}$$

which has solution

$$\frac{F_1}{F_3}=\frac{\cos\theta}{\sin\theta}, \tag{3.81}$$

up to an irrelevant constant which may be eliminated by a shift of θ. Using this, we find that

$$\partial_\theta\log\left(\frac{F_1F_2^{2/3}F_3}{\sin 2\theta}\right)=0, \tag{3.82}$$

and hence that

$$F_1F_2^{2/3}F_3=\alpha\sin 2\theta, \tag{3.83}$$

for constant α. Finaly we get

$$\partial_\theta\left(\frac{\sin 2\theta}{F_2^{2/3}}\right)=\frac{1}{\alpha}F_2^2, \tag{3.84}$$

which has solution

$$F_2=\left(\frac{3\alpha}{8}\right)^{3/8}\left[\frac{\sin 2\theta}{(\beta+[2+\cos 2\theta]\sin^4\theta)^{1/4}}\right]^{3/2}, \tag{3.85}$$

for constant β. As was the case for the three-fold solution of the previous subsection, the metric has pathological behaviour unless $\beta=0$. Choosing this value, the metric, up to an overall scale of $(8\alpha^3/3)^{1/4}$, is as given above.

4 Sp(2) interpolating pair

In this section, we will give a conjectured interpolating pair for fivebranes wrapped on a complex lagarangian four-cycle in an $Sp(2)$ manifold. First we give the pair, then the derivation of the $Sp(2)$ interpolation from the AdS solution.

The interpolating pair In [25], an AdS_3 solution admitting six Killing spinors and describing the near-horizon limit of fivebranes wrapped on a CLAG four-cycle in an $Sp(2)$ manifold was constructed. In addition to the fivebranes, there are membranes extended in the directions transverse to the $Sp(2)$, which intersect the fivebranes in a string. The quantum dual of the AdS solution is the two-dimensional low energy effective theory on the string worldvolume. The metric of the AdS solution is given by

$$\mathrm{d}s^2 = \frac{1}{\lambda}\left[\mathrm{d}s^2(AdS_3) + \frac{5}{2}\mathrm{d}s^2(\mathrm{B}_4) + (1-\lambda^3 f^2)\mathrm{D}Y^a\mathrm{D}Y^a + \frac{\lambda^3}{4(1-\lambda^3 f^2)}\mathrm{d}\rho^2\right],$$

$$\lambda^3 = \frac{50}{60+3\rho^2}, \quad f = \frac{3\rho}{5}. \tag{4.1}$$

Here $\mathrm{d}s^2(\mathrm{B}_4)$ is the Bergman metric on two-dimensional complex hyperbolic space, normalised such that the scalar curvature is $R=-12$; explicitly, this metric is

$$\mathrm{d}s^2(\mathrm{B}_4) = 2\left[\mathrm{d}z^2 + \frac{1}{4}\sinh^2 z(\sigma_1^2+\sigma_2^2+\cosh^2 z\sigma_3^2)\right], \tag{4.2}$$

with $\mathrm{d}\sigma_1 = \sigma_2\wedge\sigma_3$, together with cyclic permutations. In the AdS metric (4.1), the Y^a, $a=1,...,4$ parameterise an S^3, $Y^aY^a=1$, and

$$\mathrm{D}Y^a = \mathrm{d}Y^a + \omega^a{}_bY^b, \tag{4.3}$$

with ω_{ab} the spin connection one-forms of B_4. The electric flux is irrelevant to the discussion, and may be obtained from [25] or [41]; the magnetic flux will be given below.

To give the conjectured special holonomy interpolation of this metric, we first make the following definitions. Let e^a denote a basis for the Bergman metric (4.2). Let J^A, $A=1,2,3$, denote a basis of self-dual $SU(2)$ invariant three-forms on B_4, obeying the algebra $J^AJ^B = -\delta^{AB} - \epsilon^{ABC}J^C$, and let J^3 be the Kähler form of B_4. Define

$$\mathrm{E}_1 = J^1_{ab}Y^ae^b, \;\; \mathrm{E}_2 = -J^2_{ab}Y^ae^b, \;\; \mathrm{E}_3 = J^1_{ab}Y^a\mathrm{D}Y^b, \;\; \mathrm{E}_4 = J^2_{ab}Y^a\mathrm{D}Y^b,$$

$$\mathrm{E}_5 = J^3_{ab}Y^ae^b, \;\; \mathrm{E}_6 = Y^ae^a, \;\; \mathrm{E}_7 = J^3_{ab}Y^a\mathrm{D}Y^b. \tag{4.4}$$

Then the conjectured hyper-Kähler interpolation of the AdS_3 solution is

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathbb{R}^{1,2}) + \mathrm{d}s^2(\mathcal{N}_\tau), \tag{4.5}$$

where, up to an overall scale,

$$\begin{aligned} \mathrm{d}s^2(\mathcal{N}_\tau) &= \Big(1+R^2\Big)\Big(\mathrm{E}_1^2+\mathrm{E}_2^2\Big) + 2\Big(1-R^2\Big)\Big(\mathrm{E}_3^2+\mathrm{E}_4^2\Big) + 2R^2\Big(\mathrm{E}_5^2+\mathrm{E}_6^2\Big) \\ &\quad + R^2\left(\frac{1}{R^4}-1\right)\mathrm{E}_7^2 + 4\left(\frac{1}{R^4}-1\right)^{-1}\mathrm{d}R^2. \end{aligned} \tag{4.6}$$

The range of R is $R \in (0,1]$. At $R=1$, the S^3 degenerates smoothly. Defining $R = 1-y/2$, the metric near $y=0$ is

$$\mathrm{d}s^2(\mathcal{N}_\tau) = 2\mathrm{d}s^2(\mathrm{B}_4) + \mathrm{d}y^2 + y^2 \mathrm{D}Y^a \mathrm{D}Y^a. \tag{4.7}$$

The metric is singular at $R=0$. This $\mathcal{N}_\tau$ metric is the hyperbolic analogue of the Calabi metric on $T^*\mathbb{CP}^2$ [54]. Now we give its derivation from the AdS solution.

The G-structure of the AdS solution The AdS_3 admits an $SU(2)$ structure defined by all six Killing spinors. This structure satisfies the torsion conditions of [41], for the near-horizon limit of fivebranes on a CLAG four-cycle, together with the Bianchi identity for the flux $\mathrm{d}F_{\mathrm{mag}} = 0$, which in this case is not implied by the torsion conditions. The $SU(2)$ structure is given by

$$e^5 = \sqrt{\frac{5}{2\lambda}}\mathrm{E}_5, \quad e^6 = \sqrt{\frac{5}{2\lambda}}\mathrm{E}_6, \quad e^7 = \sqrt{\frac{1-\lambda^3 f^2}{\lambda}}E_7,$$

$$\begin{aligned} K^1 &= \frac{1}{\lambda}\sqrt{\frac{5(1-\lambda^3 f^2)}{2}}\Big(\mathrm{E}_1\wedge\mathrm{E}_4 + \mathrm{E}_2\wedge\mathrm{E}_3\Big), \\ K^2 &= \frac{1}{\lambda}\sqrt{\frac{5(1-\lambda^3 f^2)}{2}}\Big(-\mathrm{E}_1\wedge\mathrm{E}_3 + \mathrm{E}_2\wedge\mathrm{E}_4\Big), \\ K^3 &= \frac{5}{2\lambda}\mathrm{E}_1\wedge\mathrm{E}_2 + \frac{(1-\lambda^3 f^2)}{\lambda}\mathrm{E}_3\wedge\mathrm{E}_4. \end{aligned} \tag{4.8}$$

The triplet of $SU(2)$ structure forms K^A (not to be confused with the J^A forms on B_4) obey the algebra $K^A K^B = -\delta^{AB} - \epsilon^{ABC}K^C$. The relevant torsion conditions of [41] are

$$\hat{\rho}\wedge\mathrm{d}\Big[\lambda^{-1}\left(\mathrm{Vol}[\mathcal{M}_4] + K^3\wedge e^{56}\right)\Big] = 0, \tag{4.9}$$

$$(K^3 + e^{56})\lrcorner\,\mathrm{d}e^7 = \frac{2\lambda^{1/2}}{\sqrt{1-\lambda^3 f^2}}(1+\lambda^3 f^2) - \lambda^{3/2} f\hat{\rho}\lrcorner\,\mathrm{d}\log\left(\frac{\lambda^3 f}{1-\lambda^3 f^2}\right),$$

$$\mathrm{d}\left(\lambda^{-1}\sqrt{1-\lambda^3 f^2}e^5\right) = \lambda^{-1/2}\Big(K^1 + e^{67} + \lambda^{3/2} f e^5\wedge\hat{\rho}\Big), \tag{4.10}$$

$$\mathrm{d}\left(\lambda^{-1}\sqrt{1-\lambda^3 f^2}e^6\right) = \lambda^{-1/2}\Big(K^2 + e^{75} + \lambda^{3/2} f e^2\wedge\hat{\rho}\Big), \tag{4.11}$$

with

$$\mathrm{Vol}[\mathcal{M}_4] = \frac{1}{2} K^3 \wedge K^3. \tag{4.12}$$

The magnetic flux is

$$\begin{aligned} F_{\mathrm{mag}} &= \frac{\lambda^{3/2}}{1-\lambda^3 f^2}(\lambda^{3/2} f + \star_8)\Big[\mathrm{d}\left(\lambda^{-3/2}\sqrt{1-\lambda^3 f^2}\Big[K^3 \wedge e^7 + e^{567}\Big]\right) \\ &\quad -4\lambda^{-1}\Big(\mathrm{Vol}[\mathcal{M}_4] + K^3 \wedge e^{56}\Big)\Big] + 2\lambda^{1/2}\Big(K^3 \wedge e^7 + e^{567}\Big) \wedge \hat{\rho}, \end{aligned} \tag{4.13}$$

with

$$\mathrm{Vol}[\mathcal{M}_8] = \mathrm{Vol}[\mathcal{M}_4] \wedge e^{567} \wedge \hat{\rho}. \tag{4.14}$$

In verifying that the given structure indeed solves the torsion conditions and Bianchi identity, and in the derivation of the $Sp(2)$ metric to follow, the following is useful. Defining

$$Q = \frac{1}{2} J^{3ab} \omega_{ab}, \tag{4.15}$$

the exterior derivatives of the Es are given by

$$\begin{aligned} \mathrm{dE}_1 &= -\mathrm{E}_2 \wedge (Q + \mathrm{E}_7) - \mathrm{E}_3 \wedge \mathrm{E}_6 + \mathrm{E}_4 \wedge \mathrm{E}_5, \\ \mathrm{dE}_2 &= \mathrm{E}^1 \wedge (Q + \mathrm{E}_7) + \mathrm{E}_3 \wedge \mathrm{E}_5 + \mathrm{E}_4 \wedge \mathrm{E}_6, \\ \mathrm{dE}_3 &= \mathrm{E}_4 \wedge (Q + 2\mathrm{E}_7) - \frac{1}{2}\mathrm{E}_1 \wedge \mathrm{E}_6 + \frac{1}{2}\mathrm{E}_2 \wedge \mathrm{E}_5, \\ \mathrm{dE}_4 &= -\mathrm{E}_3 \wedge (Q + 2\mathrm{E}_7) + \frac{1}{2}\mathrm{E}_2 \wedge \mathrm{E}_6 + \frac{1}{2}\mathrm{E}_1 \wedge \mathrm{E}_5, \\ \mathrm{dE}_5 &= \mathrm{E}_1 \wedge \mathrm{E}_4 + \mathrm{E}_2 \wedge \mathrm{E}_3 + \mathrm{E}_6 \wedge \mathrm{E}_7, \\ \mathrm{dE}_6 &= -\mathrm{E}_1 \wedge \mathrm{E}_3 + \mathrm{E}_2 \wedge \mathrm{E}_4 + \mathrm{E}_7 \wedge \mathrm{E}_5, \\ \mathrm{dE}_7 &= -\mathrm{E}_1 \wedge \mathrm{E}_2 + 2\mathrm{E}_3 \wedge \mathrm{E}_4 - 2\mathrm{E}_5 \wedge \mathrm{E}_6. \end{aligned} \tag{4.16}$$

The AdS solution in the Minkowski frame We now use section 2 to frame-rotate the AdS solution. Defining the coordinates

$$\begin{aligned} t &= -\frac{1}{2} e^{-6r/5} \rho, \\ u &= -\sqrt{\frac{12 - 3\rho^2}{10}} e^{-r}, \end{aligned} \tag{4.17}$$

the one-forms e^8, e^9 in the Minkowski frame are given by

$$\begin{aligned} e^8 &= \lambda e^r \mathrm{d}u, \\ e^9 &= \lambda e^{6r/5} \mathrm{d}t, \end{aligned} \tag{4.18}$$

and the AdS metric in the Minkowski frame takes the form

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{1}{H_{M5}^{1/3}H_{M2}^{2/3}}\mathrm{d}s^2(\mathbb{R}^{1,1}) + \frac{H_{M5}^{2/3}}{H_{M2}^{2/3}}\mathrm{d}t^2 + \frac{H_{M2}^{1/3}}{H_{M5}^{1/3}}\left[\frac{5}{2}F\mathrm{d}s^2(\mathrm{B}_4)\right] \\
&\quad + H_{M2}^{1/3}H_{M5}^{2/3}\left[\frac{1}{F}\left(\mathrm{d}u^2 + u^2\mathrm{D}Y^a\mathrm{D}Y^a\right)\right],
\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}
H_{M5} &= \lambda^3 e^{22r/5}, \\
H_{M2} &= e^{4r/5}, \\
F &= e^{6r/5}.
\end{aligned} \tag{4.20}$$

The function e^{2r} is given in terms of t and u by a positive signature metric inducing root of the sextic

$$t^2e^{12r} - \left(1 - \frac{5}{6}u^2e^{2r}\right)^5 = 0. \tag{4.21}$$

The wrapped brane $Sp(2)$ structure of the AdS_3 solution, defined by three of its Killing spinors, is given by

$$\begin{aligned}
J^1 &= K^3 + e^5\wedge e^6 + e^7\wedge e^8, \\
J^2 &= K^2 - e^5\wedge e^7 + e^6\wedge e^8, \\
J^3 &= K^1 + e^6\wedge e^7 + e^5\wedge e^8.
\end{aligned} \tag{4.22}$$

By construction, this structure is a solution of the wrapped brane equations for a CLAG four-cycle in a hyper-Kähler eight-manifold, which comprise the torsion conditions [41]

$$\begin{aligned}
\mathrm{d}(L^{-1/2}J^2) = \mathrm{d}(L^{-1/2}J^3) &= 0, \\
e^9\wedge[J^1\lrcorner\,\mathrm{d}J^1 - Le^9\lrcorner\,\mathrm{d}(L^{-1}e^9)] &= 0,
\end{aligned} \tag{4.23}$$

together with the Bianchi identity and field equation for the four-form, which is given in [41].

The conjectured hyper-Kähler interpolation We make the following ansatz for an interpolating solution:

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{1}{H_{M5}^{1/3}H_{M2}^{2/3}}\mathrm{d}s^2(\mathbb{R}^{1,1}) + \frac{H_{M5}^{2/3}}{H_{M2}^{2/3}}\mathrm{d}t^2 + \frac{H_{M2}^{1/3}}{H_{M5}^{1/3}}\left[F_1^2\Big(\mathrm{E}_1^2+\mathrm{E}_2^2\Big) + F_2^2\Big(\mathrm{E}_5^2+\mathrm{E}_6^2\Big)\right] \\
&\quad + H_{M2}^{1/3}H_{M5}^{2/3}\left[F_5^2\mathrm{d}u^2 + F_3^2\Big(\mathrm{E}_3^2+\mathrm{E}_4^2\Big) + F_4^2\mathrm{E}_7^2\right],
\end{aligned} \tag{4.24}$$

with $H_{M5,M2}$, $F_{1,\ldots,5}$ arbitrary functions of u, t. To determine the hyper-Kähler interpolation with this ansatz, we set $H_{M5,M2} = 1$ and require that $F_{1,\ldots,5}$ are functions only of u. Then F_5 is at our disposal, and we set it to 1. Requiring $Sp(2)$ holonomy, we set

$$\mathrm{d}J^A = 0. \tag{4.25}$$

From $\mathrm{d}J^1$, we derive the conditions

$$\begin{aligned} \partial_u \left(F_1^2\right) &= F_4, \\ \partial_u \left(F_2^2\right) &= 2F_4, \\ \partial_u \left(F_3^2\right) &= -2F_4, \\ F_1^2 &= F_2^2 + \frac{1}{2}F_3^2. \end{aligned} \tag{4.26}$$

The algebraic constraint, combined with any two of the differential equations, implies the third. From $\mathrm{d}J^2$, we get

$$\begin{aligned} \partial_u \left(F_1 F_3\right) &= -F_2, \\ \partial_u \left(F_2 F_4\right) &= -F_2, \\ F_1 F_3 &= F_2 F_4, \end{aligned} \tag{4.27}$$

and from $\mathrm{d}J^3$ we again obtain the equations (4.27). The algebraic constraint in (4.27), combined with either of the differential equations, implies the second. Therefore the system we need to solve is

$$\begin{aligned} \partial_u \left(F_1^2\right) &= F_4, \\ \partial_u \left(F_2^2\right) &= 2F_4, \\ \partial_u \left(F_2 F_4\right) &= -F_2, \\ F_3^2 &= 2\left(F_1^2 - F_2^2\right), \\ F_1 F_3 &= F_2 F_4. \end{aligned} \tag{4.28}$$

To solve the system, define a new coordinate x such that

$$\partial_u = F_4 \partial_x. \tag{4.29}$$

Then the first two equations of (4.28) give

$$\begin{aligned} F_1^2 &= x + a, \\ F_2^2 &= 2x + b, \end{aligned} \tag{4.30}$$

for constants a, b. We eliminate b by a shift of x. Integrating the third equation we get

$$F_4^2 = \frac{c}{x} - x, \tag{4.31}$$

for a constant c. Then the algebraic conditions imply that

$$\begin{aligned} F_3^2 &= 2(a-x), \\ c &= a^2. \end{aligned} \tag{4.32}$$

Finally, defining a new coordinate $x = aR^2$, up to an overall scale of a, we get the hyper-Kähler $\mathcal{N}_\tau$ metric given above.

5 G_2 interpolating pairs

In this section, we will give conjectured interpolating pairs for fivebranes wrapped on calibrated cycles in G_2 manifolds. First we will discuss co-associative four-cycles, then associative three-cycles. In each case we will first give the conjectured pairs, followed by the derivation of the G_2 interpolations from the AdS solutions.

5.1 Co-associative cycles

The interpolating pairs The GKW AdS_3 solutions [24], describing the near-horizon limit of M-fivebranes wrapped on a co-associative cycle in a manifold of G_2 holonomy, admit four Killing spinors, and have metrics

$$\mathrm{d}s^2 = \frac{1}{\lambda}\left[\mathrm{d}s^2(AdS_3) + \frac{9}{4}\mathrm{d}s^2(\Sigma_4) + \frac{9}{4}(1-\lambda^3\rho^2)\mathrm{D}Y^a\mathrm{D}Y^a + \frac{\lambda^3}{4}\left(\frac{\mathrm{d}\rho^2}{1-\lambda^3\rho^2} + \rho^2\mathrm{d}s^2(S^1)\right)\right],$$

$$\lambda^3 = \frac{81}{64+54\rho^2}. \tag{5.1}$$

In this case the flux is purely magnetic, and is irrelevant to the discussion; it may be obtained from [24] or [39]. The wrapped cycle Σ_4 is an arbitrary conformally half-flat Einstein manifold, with scalar curvature normalised such that $R = -12$. This means that the Ricci tensor of Σ_4 is given by

$$R_{ij} = -3g_{ij}, \tag{5.2}$$

and the Weyl tensor is anti-self-dual,

$$J_4^{aij} C_{ijkl} = 0, \tag{5.3}$$

for a triplet of self-dual two-forms J_4^a, $a = 1, 2, 3$, on Σ_4. An example of such a manifold is hyperbolic four-space H^4. The Y^a are constrained coordinates on S^2, $Y^aY^a = 1$, and

$$\mathrm{D}Y^a = \mathrm{d}Y^a - \frac{1}{2}\epsilon^{abc}Y^b\omega_{ij}J_4^{cij}, \tag{5.4}$$

where ω_{ij} are the spin connection one-forms of Σ_4. The range of ρ, which without loss of generality is taken to be non-negative, is $\rho \in [0, 8/3\sqrt{3}]$. At $\rho = 0$ the R-symmetry S^1 degenerates smoothly[9], while at $\rho = 8/3\sqrt{3}$ the S^2 parameterised by the Y^a degenerates smoothly.

The metric of the conjectured G_2 interpolation of these AdS solutions is

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathbb{R}^{1,3}) + \mathrm{d}s^2(\mathcal{N}_\tau), \tag{5.5}$$

where up to an overall scale,

$$\mathrm{d}s^2(\mathcal{N}_\tau) = \frac{R^2}{2}\mathrm{d}s^2(\Sigma_4) + \frac{R^2}{4}\left(\frac{1}{R^4} - 1\right)\mathrm{D}Y^a\mathrm{D}Y^a + \left(\frac{1}{R^4} - 1\right)^{-1}\mathrm{d}R^2. \tag{5.6}$$

The range of R is $R \in (0, 1]$. At $R = 1$, the S^2 degenerates smoothly. The metric is singular at $R = 0$ where the co-associative Σ_4 degenerates. These metrics are the analogues, for negatively curved conformally half-flat Einstein Σ_4, of the regular BSGPP G_2 metrics on $\mathbb{R}^3$ bundles over S^4 or $\mathbb{CP}^2$ [43], [44]. Now we give their derivation from the AdS solutions.

The G-structure of the AdS solutions The $SU(3)$ structure of the AdS solutions, defined by all four of their Killing spinors, is given by [39]

$$\begin{aligned} J_6 &= \frac{9}{4\lambda}Y^aJ_4^a + \frac{9}{4\lambda}(1 - \lambda^3\rho^2)\frac{1}{2}\epsilon^{abc}Y^a\mathrm{D}Y^b \wedge \mathrm{D}Y^c, \\ \Omega_6 &= \frac{27}{8}\sqrt{\frac{1 - \lambda^3\rho^2}{\lambda^3}}(\epsilon^{abc}Y^a\mathrm{D}Y^b \wedge J_4^c + i\mathrm{D}Y^a \wedge J_4^a). \end{aligned} \tag{5.7}$$

This structure is a solution of the AdS torsion conditions of [39] for the near-horizon limit of fivebranes on a co-associative four-cycle, which are

$$\begin{aligned} \mathrm{d}\left(\frac{1}{\lambda^{3/2}\rho}J_6 \wedge \hat{\rho} - \mathrm{Im}\Omega_6\right) &= 0, \\ \mathrm{d}\left(\frac{1}{2\lambda}J_6 \wedge J_6 + \lambda^{1/2}\rho\mathrm{Re}\Omega_6 \wedge \hat{\rho}\right) &= 0. \end{aligned} \tag{5.8}$$

[9]The R-symmetry of the conformal duals is $U(1)$.

The following identities, valid for an arbitrary conformally half-flat Einstein manifold of scalar curvature R, are useful in verifying this claim:

$$\begin{aligned}
\mathrm{d}(Y^a J_4^a) &= \mathrm{D}Y^a \wedge J_4^a, \\
\mathrm{d}\left(\frac{1}{2}\epsilon^{abc} Y^a \mathrm{D}Y^b \wedge \mathrm{D}Y^c\right) &= \frac{R}{12}\mathrm{D}Y^a \wedge J_4^a, \\
\mathrm{d}(\epsilon^{abc} Y^a \mathrm{D}Y^b \wedge J_4^c) &= \frac{R}{3}\mathrm{Vol}[\Sigma_4] + Y^d J_4^d \wedge \epsilon^{abc} Y^a \mathrm{D}Y^b \wedge \mathrm{D}Y^c.
\end{aligned} \tag{5.9}$$

In this case the Bianchi identity for the four-form is implied by the torsion conditions [39].

The AdS solution in the Minkowski frame Using section 2, we now frame-rotate these solutions to the canonical Minkowski frame. The one-form $\hat{u}$ is given by

$$\hat{u} = Le^{-4r/3}\mathrm{d}\left(-\frac{1}{6}\sqrt{64 - 27\rho^2}e^{-2r/3}\right). \tag{5.10}$$

Defining the Minkowski frame coordinate u,

$$u = -\frac{1}{6}\sqrt{64 - 27\rho^2}e^{-2r/3}, \tag{5.11}$$

the AdS_3 solutions in the Minkowski frame are given by

$$\mathrm{d}s^2 + L^{-1}\left[\mathrm{d}s^2(\mathbb{R}^{1,1}) + \frac{9}{4}F\mathrm{d}s^2(\Sigma_4)\right] + L^2\Big[F^{-4/3}\left(\mathrm{d}u^2 + u^2\mathrm{D}Y^a\mathrm{D}Y^a\right) + \mathrm{d}s^2(\mathbb{R}^2)\Big], \tag{5.12}$$

where

$$\begin{aligned}
F &= e^{2r}, \\
L &= \lambda F,
\end{aligned} \tag{5.13}$$

and e^{4r} is a positive signature metric inducing root of the cubic

$$\left(\frac{16}{9} - t^2 e^{4r}\right)^3 - u^6 e^{4r} = 0. \tag{5.14}$$

The wrapped-brane G_2 structure of the AdS_3 solutions is defined by two of their Killing spinors, and is given by

$$\begin{aligned}
\Phi &= J_6 \wedge \hat{u} - \mathrm{Im}\Omega_6, \\
\Upsilon &= \frac{1}{2}J_6 \wedge J_6 + \mathrm{Re}\Omega_6 \wedge \hat{u}.
\end{aligned} \tag{5.15}$$

By construction, this structure is a solution of the wrapped brane equations for fivebranes on a co-associative four-cycle. From [57], [39], these equations are

$$
\begin{aligned}
\mathrm{Vol}[\mathbb{R}^2]\wedge \mathrm{d}\Phi &= 0,\\
\mathrm{d}(L^{-1}\Phi\wedge\Upsilon) &= 0,\\
\Phi\wedge\mathrm{d}\Phi &= 0,\\
\mathrm{d}\Big(L\star_9\mathrm{d}(L^{-1}\Upsilon)\Big) &= 0.
\end{aligned}
\tag{5.16}
$$

In the last equation, which comes from the four-form Bianchi identity, $\star_9$ denotes the Hodge dual on the space transverse to the Minkowski factor.

The conjectured G_2 interpolation We now make the following ansatz for an interpolating solution:

$$
\mathrm{d}s^2+L^{-1}\left[\mathrm{d}s^2(\mathbb{R}^{1,1})+F_1^2\mathrm{d}s^2(\Sigma_4)\right]+L^2\left[F_3^2\mathrm{d}u^2+F_2^2\mathrm{D}Y^a\mathrm{D}Y^a+\mathrm{d}s^2(\mathbb{R}^2)\right], \tag{5.17}
$$

with $L, F_{1,2,3}$ functions of u, t. For special holonomy we must have $L=$ constant, which we take to be unity. We also must have that $F_{1,2,3}$ are functions of u only; the function F_3 is then at our disposal, and we set it to 1. The condition of G_2 holonomy is then

$$
\mathrm{d}\Phi=\mathrm{d}\Upsilon=0, \tag{5.18}
$$

for the metric

$$
\mathrm{d}s^2(\mathcal{N}_\tau)=F_1^2\mathrm{d}s^2(\Sigma_4)+F_2^2\mathrm{D}Y^a\mathrm{D}Y^a+\mathrm{d}u^2, \tag{5.19}
$$

with the G_2 structure inherited from the AdS frame,

$$
\begin{aligned}
\Phi &= J_6\wedge\mathrm{d}u-\mathrm{Im}\Omega_6,\\
\Upsilon &= \frac{1}{2}J_6\wedge J_6+\mathrm{Re}\Omega\wedge\mathrm{d}u,\\
J_6 &= F_1^2Y^aJ_4^a+\frac{1}{2}F_2^2\epsilon^{abc}Y^a\mathrm{D}Y^b\wedge\mathrm{D}Y^c,\\
\Omega_6 &= F_1^2F_2(\epsilon^{abc}Y^a\mathrm{D}Y^b\wedge J_4^c+i\mathrm{D}Y^a\wedge J_4^a).
\end{aligned}
\tag{5.20}
$$

With $R=-12$, closure of Φ implies

$$
\partial_u(F_1^2F_2)=F_2^2-F_1^2, \tag{5.21}
$$

while closure of Υ implies

$$
\begin{aligned}
\partial_u(F_1^4) &= 4F_1^2F_2,\\
2\partial_u(F_1^2F_2^2) &= -4F_1^2F_2.
\end{aligned}
\tag{5.22}
$$

It is readily verified that (5.22) imply (5.21). Integrating (5.22) is straightforward. Adding, we find that

$$F_2^2 = \frac{\alpha^2}{2F_1^2} - \frac{F_1^2}{2}, \tag{5.23}$$

for some constant α. Defining a new coordinate x such that

$$\partial_u = 4F_1^2 F_2 \partial_x, \tag{5.24}$$

we then get

$$F_1^4 \;\; = \;\; x + \beta, \tag{5.25}$$

for an irrelevant constant β which can be eliminated by a shift in x. The constant α^2 may be set to unity, up to an overall scale in the metric. Defining a new coordinate $R^4 = x/4$, the G_2 metrics conjectured to be the interpolation of the co-associative AdS_3 solutions of [24] are as given above.

5.2 Associative cycle

The interpolating pair The AdS_4 solution of [22], describing the near-horizon limit of M-fivebranes wrapped on an associative three-cycle in a G_2 manifold, admits four Killing spinors, and is as follows. The metric is given by

$$\mathrm{d}s^2 = \frac{1}{\lambda}\left[\mathrm{d}s^2(AdS_4) + \frac{4}{5}\mathrm{d}s^2(H^3) + \frac{4}{25}(1-\lambda^3\rho^2)\mu^a\mu^a + \frac{\lambda^3}{4}\frac{\mathrm{d}\rho^2}{1-\lambda^3\rho^2}\right],$$

$$\lambda^3 = \frac{8}{5+3\rho^2}. \tag{5.26}$$

The flux is purely magnetic and is irrelevant to the discussion; it may be obtained from [22] or [39]. The μ^a, $a = 1, 2, 3$, are given by

$$\mu^a = \sigma^a - \frac{1}{2}\epsilon^{abc}\omega_{ab}, \tag{5.27}$$

where the σ^a are left-invariant one-foms on an S^3, $\mathrm{d}\sigma^a = \frac{1}{2}\epsilon^{abc}\sigma^b \wedge \sigma^c$, and the ω_{ab} are the spin-connection one-forms of H^3. The range of ρ is $\rho \in [-1, 1]$, with the S^3 degenerating smoothly at $\rho = \pm 1$.

The conjectured G_2 interpolation of this metric is

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathbb{R}^{1,3}) + \mathrm{d}s^2(\mathcal{N}_\tau), \tag{5.28}$$

where up to an overall scale

$$\mathrm{d}s^2(\mathcal{N}_\tau) = \frac{R^2}{3}\mathrm{d}s^2(H^3) + \frac{R^2}{9}\Big(\frac{1}{R^3} - 1\Big)\mu^a\mu^a + \left(\frac{1}{R^3} - 1\right)^{-1}\mathrm{d}R^2. \tag{5.29}$$

This metric is singular where the associative H^3 degenerates, at $R = 0$. At $R = 1$, the S^3 degenerates smoothly. This $\mathcal{N}_\tau$ metric is a singular hyperbolic analogue of the BSGPP G_2 metric on an $\mathbb{R}^4$ bundle over S^3 of [43], [44]. This $\mathcal{N}_\tau$ metric was also found in [59], as the conjectured G_2 interpolation of the AdS_2 IIB solution of [26], for D3 branes wrapped on an associative three-cycle. If it is indeed the interpolation of both these AdS solutions, then there are two distinct conformal theories that have their origins in this geometry. The first is a superconformal quatum mechanics, arising on the unwrapped (time) direction of D3-branes on the H^3 of (5.29); the second is a three-dimensional superconformal theory, arising on the unwrapped worldvolume directions of M5-branes on the H^3. Now we discuss the derivation of $\mathcal{N}_\tau$ from the M-theory AdS_4 solution.

The G-structure of the AdS solution With e^a a basis for H^3, the $SU(3)$ structure of the AdS solution, defined by all its four Killing spinors, is [39]

$$\begin{aligned}
J_6 &= \frac{4}{5\sqrt{5}\lambda}\sqrt{1-\lambda^3\rho^2}\mu^a\wedge e^a,\\
\mathrm{Im}\Omega_6 &= \frac{8}{25\sqrt{5}\lambda^{3/2}}(1-\lambda^3\rho^2)\frac{1}{2}\epsilon^{abc}e^a\wedge\mu^b\wedge\mu^c - \frac{8}{5\sqrt{5}\lambda^{3/2}}\mathrm{Vol}[H^3],\\
\mathrm{Re}\Omega_6 &= \left(\frac{2}{5\lambda^{1/2}}\sqrt{1-\lambda^3\rho^2}\right)^3\frac{1}{3!}\epsilon^{abc}\mu^a\wedge\mu^b\wedge\mu^c - \frac{8}{25}\sqrt{\frac{1-\lambda^3\rho^2}{\lambda^3}}\frac{1}{2}\epsilon^{abc}\mu^a\wedge e^b\wedge e^c.
\end{aligned}\tag{5.30}$$

This structure is a solution of the AdS torsion conditions of [58], interpreted in [39] as the conditions defining the near-horizon limit of fivebranes wrapped on an associative three-cycle, which are

$$\begin{aligned}
\mathrm{d}\left(\rho J_6\wedge\hat{\rho} - \frac{1}{\lambda^{3/2}}\mathrm{Im}\Omega_6\right) &= 0,\\
\mathrm{d}\left(\frac{1}{2\lambda\rho}J_6\wedge J_6 + \frac{1}{\lambda^{5/2}\rho^2}\mathrm{Re}\Omega\wedge\hat{\rho}\right) &= 0.
\end{aligned}\tag{5.31}$$

Some useful identities in verifying this claim are

$$\begin{aligned}
\mathrm{d}(\mu^a\wedge e^a) &= \frac{1}{2}\epsilon^{abc}\mu^a\wedge\mu^b\wedge e^c + 3\mathrm{Vol}[H^3],\\
\mathrm{d}\left(\frac{1}{3!}\epsilon^{abc}\mu^a\wedge\mu^b\wedge\mu^c\right) = \mathrm{d}\left(\frac{1}{2}\epsilon^{abc}\mu^a\wedge e^b\wedge e^c\right) &= \frac{1}{2}e^a\wedge e^b\wedge\mu^a\wedge\mu^b.
\end{aligned}\tag{5.32}$$

The AdS solution in the Minkowski frame Now we use section 2 to frame-rotate to the canonical Minkowski frame. The one-form $\hat{u}$ is given by

$$\hat{u} = Le^{-3r/4}\mathrm{d}u, \tag{5.33}$$

with the Minkowski-frame coordinate u given by

$$u = -\frac{4}{5}\sqrt{\frac{5-5\rho^2}{8}}. \tag{5.34}$$

Then the associative AdS_4 solution in the Minkowski frame is

$$\mathrm{d}s^2 = L^{-1}\left[\mathrm{d}s^2(\mathbb{R}^{1,2}) + \frac{4}{5}e^{2r}\mathrm{d}s^2(H^3)\right] + L^2\left[F^{-3/4}\left(\mathrm{d}u^2 + \frac{u^2}{4}\mu^a\mu^a\right) + \mathrm{d}t^2\right], \tag{5.35}$$

where

$$\begin{aligned} F &= e^{2r}, \\ L &= \lambda F, \end{aligned} \tag{5.36}$$

and e^r is a positive signature metric inducing root of the octic

$$\frac{4}{25}(1-4t^2e^{4r})^2 - u^2e^{5r} = 0. \tag{5.37}$$

The wrapped-brane G_2 structure of the associative AdS_4 solution, defined by two of its Killing spinors, is given by

$$\begin{aligned} \Phi &= J_6\wedge\hat{u} - \mathrm{Im}\Omega_6, \\ \Upsilon &= \frac{1}{2}J_6\wedge J_6 + \mathrm{Re}\Omega_6\wedge\hat{u}. \end{aligned} \tag{5.38}$$

By construction, this structure is a solution of the wrapped brane equations for an associative three-cycle. From [37], [39], these are

$$\begin{aligned} \mathrm{d}t\wedge\mathrm{d}(L^{-1}\Upsilon) &= 0, \\ \mathrm{d}(L^{-5/2}\Phi\wedge\Upsilon) &= 0, \\ \Phi\wedge\mathrm{d}\Phi &= 0, \\ \mathrm{d}\Big(L^{3/2}\star_8\mathrm{d}(L^{-3/2}\Phi)\Big) &= 0, \end{aligned} \tag{5.39}$$

where in the last equation (the four-form Bianchi identity), $\star_8$ denotes the Hodge dual on the space transverse to the Minkowski factor.

The conjectured G_2 interpolation We now conjecture the existence of a solution of (5.39) which smoothly interpolates between (5.35) and a manifold of G_2 holonomy. We make the following metric ansatz for this solution:

$$\mathrm{d}s^2 = L^{-1}\left[\mathrm{d}s^2(\mathbb{R}^{1,2}) + F_1^2\mathrm{d}s^2(H^3)\right] + L^2\left[F_3^2\mathrm{d}u^2 + F_2^2\mu^a\mu^a + \mathrm{d}t^2\right], \tag{5.40}$$

with L, $F_{1,2,3}$ functions of u and t. For special holonomy we set $L = 1$, and require that $F_{1,2,3}$ are arbitrary functions of u. In fact, the determination of the G_2 metric from this point on exactly follows that of [59], where a conjectured G_2 interpolation of the AdS_2 solution of [26] for a D3 brane wrapped on an associative three-cycle was studied. The ansatz for the G_2 manifold is exactly the same, and the reader is referred to [59] for the rest of the derivation, or invited to perform it as a useful excercise.

6 Spin(7) interpolating pairs

In this section, we will give conjectured interpolating pairs for fivebranes wrapped on Cayley four-cycles in $Spin(7)$ manifolds. First we give the pairs, then the derivation of the $Spin(7)$ interpolations.

The interpolating pairs The GKW AdS_3 solutions [24] describing the near-horizon limit of fivebranes on Cayley four-cycles, with membranes in the overall transverse directions, admit two Killing spinors and have metrics given by

$$\mathrm{d}s^2 = \frac{1}{\lambda}\left[\mathrm{d}s^2(AdS_3) + \frac{7}{4}\mathrm{d}s^2(\Sigma_4) + (1-\lambda^3 f^2)\mathrm{D}Y^a\mathrm{D}Y^a + \frac{\lambda^3}{4(1-\lambda^3 f^2)}\mathrm{d}\rho^2\right],$$

$$\lambda^3 = \frac{49}{84+15\rho^2}, \quad f = \frac{6\rho}{7}. \tag{6.1}$$

The electric flux may be obtained from [24] or [41], and the magnetic flux will be given below. The wrapped cycle Σ_4 is an arbitrary conformally-half flat negative scalar curvature Einstein four-manifold, normalised such that the Ricci scalar is $R = -12$. We have flipped the definition of orientation on Σ_4 relative to [24]; the conformally half-flat condition reads $J^{Aij}C_{ijkl} = 0$, with J^A, $A = 1, 2, 3$, a basis of self-dual two-forms on Σ_4 and C_{ijkl} the Weyl tensor on Σ_4. The Y^a, $a = 1, ..., 4$ are constrained coordinates on an S^3, $Y^aY^a = 1$, and

$$\mathrm{D}Y^a = \mathrm{d}Y^a + \frac{1}{4}\omega_{cd}J^{Acd}J^{Aa}{}_bY^b, \tag{6.2}$$

where ω_{ab} are the spin connection one-forms of Σ_4. The range of ρ is $\rho \in [-2, 2]$; at the end-points, the S^3 degenerates smoothly.

The conjectured $Spin(7)$ interpolation of this metric is

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathbb{R}^{1,2}) + \mathrm{d}s^2(\mathcal{N}_\tau), \tag{6.3}$$

where up to an overall scale

$$\mathrm{d}s^2(\mathcal{N}_\tau) = \frac{9}{20}R^2\mathrm{d}s^2(\Sigma_4) + \frac{36}{100}R^2\left(\frac{1}{R^{10/3}} - 1\right)\mathrm{D}Y^a\mathrm{D}Y^a + \left(\frac{1}{R^{10/3}} - 1\right)^{-1}\mathrm{d}R^2. \tag{6.4}$$

These metrics are singular at $R = 0$, where the Cayley four-cycle Σ_4 degenerates. At $R = 1$, the S^3 degenerates smoothly. As discussed in the introduction these metrics are the analogues, for negatively curved conformally half-flat Einstein Σ_4, of the regular BSGPP $Spin(7)$ metric on an $\mathbb{R}^4$ bundle over S^4, [43], [44]. We now give the derivation of the $\mathcal{N}_\tau$ metric from the AdS metric.

The G-structure of the AdS solution The solution admits a G_2 structure, defined by both its Killing spinors, which satisfies the torsion conditions of [37][10] together with the Bianchi identity for the magnetic flux (also given in [37]) which in this case is not implied by the torsion conditions. the torsion conditions of [37] were interpreted in [41] as the conditions defining the near-horizon limit of fivebranes wrapped on a Cayley four-cycle. These conditions are satisfied by all supersymmetric AdS_3 solutions of M-theory. If e^a denote a basis for Σ_4, the G_2 structure of the AdS solutions is given by

$$\begin{aligned}
\Phi &= -\frac{7}{4}\sqrt{\frac{1-\lambda^3 f^2}{\lambda^3}}\left[Y^a e^a\wedge e^b\wedge \mathrm{D}Y^b + \frac{1}{2}\epsilon^{abcd}Y^a\mathrm{D}Y^b\wedge e^c\wedge e^d\right] \\
&\quad + \left(\sqrt{\frac{1-\lambda^3 f^2}{\lambda}}\right)^3\frac{1}{3!}\epsilon^{abcd}Y^a\mathrm{D}Y^b\wedge \mathrm{D}Y^c\wedge \mathrm{D}Y^d, \qquad (6.5)\\
\Upsilon &= -\frac{7}{4\lambda^2}(1-\lambda^3 f^2)\left[\frac{1}{2}e^a\wedge e^b\wedge \mathrm{D}Y^a\wedge \mathrm{D}Y^b + \frac{1}{4}\epsilon^{abcd}\mathrm{D}Y^a\wedge \mathrm{D}Y^c\wedge e^c\wedge e^d\right] + \frac{49}{16\lambda^2}\mathrm{Vol}[\Sigma_4]. \qquad (6.6)
\end{aligned}$$

The torsion conditions of [37] are

$$\begin{aligned}
\hat{\rho}\wedge \mathrm{d}(\lambda^{-1}\Upsilon) &= 0, \qquad (6.7)\\
\mathrm{d}\left(\lambda^{-5/2}\sqrt{1-\lambda^3 f^2}\mathrm{Vol}[\mathcal{M}_7]\right) &= -4\lambda^{-1/2}f\hat{\rho}\wedge \mathrm{Vol}[\mathcal{M}_7], \qquad (6.8)\\
\mathrm{d}\Phi\wedge\Phi &= \frac{4\lambda^{1/2}}{\sqrt{1-\lambda^3 f^2}}(3-\lambda^3 f^2)\mathrm{Vol}[\mathcal{M}_7] - 2\lambda^{3/2}f\star_8 \mathrm{d}\log\left(\frac{\lambda^3 f}{1-\lambda^3 f^2}\right), \qquad (6.9)
\end{aligned}$$

where

$$\mathrm{Vol}[\mathcal{M}_7] = \frac{1}{7}\Phi\wedge\Upsilon. \tag{6.10}$$

[10] The conditions of [37] contain a minor error which is corrected in [41].

The four-form Bianchi identity is $\mathrm{d}F_{\mathrm{mag}} = 0$, with

$$F_{\mathrm{mag}} \;=\; \frac{\lambda^{3/2}}{\sqrt{1-\lambda^3 f^2}}\Big(\lambda^{3/2} f + \star_8\Big)\Big(\mathrm{d}[\lambda^{-3/2}\sqrt{1-\lambda^3 f^2}\Phi] - 4\lambda^{-1}\Upsilon\Big) + 2\lambda^{1/2}\Phi\wedge\hat{\rho}, \quad (6.11)$$

where $\star_8$ denotes the Hodge dual on the space transverse to the AdS_3 factor, with positive orientation defined with respect to

$$\mathrm{Vol} = \mathrm{Vol}[\mathcal{M}_7]\wedge\hat{\rho}. \quad (6.12)$$

It may be verified that the structure (6.5) is indeed a solution of the torsion conditions and Bianchi identity, by using the following identities, valid for any conformally half-flat Einstein Σ_4 with scalar curvature R:

$$\begin{aligned}
&\mathrm{d}\left[Y^a e^a\wedge e^b\wedge \mathrm{D}Y^b + \frac{1}{2}\epsilon^{abcd}Y^a\mathrm{D}Y^b\wedge e^c\wedge e^d\right]\\
&= -\frac{R}{4}\mathrm{Vol}[\Sigma_4] + e^a\wedge e^b\wedge\mathrm{D}Y^a\wedge\mathrm{D}Y^b + \frac{1}{2}\epsilon^{abcd}\mathrm{D}Y^a\wedge\mathrm{D}Y^c\wedge e^c\wedge e^d,\\
&\mathrm{d}\left[\frac{1}{3!}\epsilon^{abcd}Y^a\mathrm{D}Y^b\wedge\mathrm{D}Y^c\wedge\mathrm{D}Y^d\right]\\
&= -\frac{R}{48}\left[e^a\wedge e^b\wedge\mathrm{D}Y^a\wedge\mathrm{D}Y^b + \frac{1}{2}\epsilon^{abcd}\mathrm{D}Y^a\wedge\mathrm{D}Y^c\wedge e^c\wedge e^d\right].
\end{aligned} \quad (6.13)$$

The AdS solutions in the Minkowski frame Defining the coordinates

$$\begin{aligned}
t &= -\frac{1}{2}e^{-12r/7}\rho,\\
u &= -\sqrt{\frac{12-3\rho^2}{7}}e^{-r},
\end{aligned} \quad (6.14)$$

the one-forms e^8, e^9 in the Minkowski frame are given by

$$\begin{aligned}
e^8 &= \lambda e^r \mathrm{d}u,\\
e^9 &= \lambda e^{12r/7}\mathrm{d}t,
\end{aligned} \quad (6.15)$$

and the metric in the Minkowski frame takes the form

$$\begin{aligned}
\mathrm{d}s^2 \;=\;& \frac{1}{H_{M5}^{1/3}H_{M2}^{2/3}}\mathrm{d}s^2(\mathbb{R}^{1,1}) + \frac{H_{M5}^{2/3}}{H_{M2}^{2/3}}\mathrm{d}t^2 + \frac{H_{M2}^{1/3}}{H_{M5}^{1/3}}\left[\frac{7}{4}F\mathrm{d}s^2(\Sigma_4)\right]\\
&+H_{M2}^{1/3}H_{M5}^{2/3}\left[\frac{1}{F}\left(\mathrm{d}u^2 + u^2\mathrm{D}Y^a\mathrm{D}Y^a\right)\right],
\end{aligned} \quad (6.16)$$

where

$$\begin{aligned}
H_{M5} &= \lambda^3 e^{38r/7},\\
H_{M2} &= e^{2r/7},\\
F &= e^{12r/7}.
\end{aligned} \quad (6.17)$$

The function e^{2r} is given in terms of t and u by a positive signature metric inducing root of the twelfth order polynomial

$$t^{14}e^{24r} - \left(1 - \frac{7}{12}u^2e^{2r}\right)^7 = 0. \tag{6.18}$$

The wrapped brane $Spin(7)$ structure of the AdS_3 solutions, defined by one of their Killing spinors, is given by

$$\phi = -\Phi \wedge e^8 - \Upsilon. \tag{6.19}$$

By construction, this structure is a solution of the wrapped brane equations for a Cayley four-cycle in a $Spin(7)$ manifold, which comprise the torsion conditions [61], [41]

$$\begin{aligned} e^9 \wedge \left[-L^3 e^9 \lrcorner \, \mathrm{d}(L^{-3}e^9) + \frac{1}{2}\phi \lrcorner \, \mathrm{d}\phi\right] &= 0, & (6.20)\\ (e^9 \wedge +\star_9)[e^9 \lrcorner \, \mathrm{d}(L^{-1}\phi)] &= 0, & (6.21) \end{aligned}$$

together with the Bianchi identity and field equation for the four-form, which is given in [61], [41].

The conjectured Spin(7) interpolation We make the following ansatz for an interpolating solution:

$$\begin{aligned} \mathrm{d}s^2 &= \frac{1}{H_{M5}^{1/3}H_{M2}^{2/3}}\mathrm{d}s^2(\mathbb{R}^{1,1}) + \frac{H_{M5}^{2/3}}{H_{M2}^{2/3}}\mathrm{d}t^2 + \frac{H_{M2}^{1/3}}{H_{M5}^{1/3}}\left[F_1^2\mathrm{d}s^2(\Sigma_4)\right] \\ &\quad + H_{M2}^{1/3}H_{M5}^{2/3}\left[F_3^2\mathrm{d}u^2 + F_2^2\mathrm{D}Y^a\mathrm{D}Y^a\right], \end{aligned} \tag{6.22}$$

with $H_{M5,M2}$, $F_{1,2,3}$ arbitrary functions of u, t. To determine the $Spin(7)$ interpolation with this ansatz, we set $H_{M5,M2} = 1$ and require that $F_{1,2,3}$ are functions only of u. Then F_3 is at our disposal, and we set it to 1. Requiring $Spin(7)$ holonomy, we set

$$\mathrm{d}\phi = 0, \tag{6.23}$$

with

$$\begin{aligned} \phi &= -\Phi \wedge \mathrm{d}u - \Upsilon, \\ \Phi &= -F_1^2F_2\left[Y^ae^a \wedge e^b \wedge \mathrm{D}Y^b + \frac{1}{2}\epsilon^{abcd}Y^a\mathrm{D}Y^b \wedge e^c \wedge e^d\right] + F_2^3\frac{1}{3!}\epsilon^{abcd}Y^a\mathrm{D}Y^b \wedge \mathrm{D}Y^c \wedge \mathrm{D}Y^d, \\ \Upsilon &= -F_1^2F_2^2\left[\frac{1}{2}e^a \wedge e^b \wedge \mathrm{D}Y^a \wedge \mathrm{D}Y^b + \frac{1}{4}\epsilon^{abcd}\mathrm{D}Y^a \wedge \mathrm{D}Y^c \wedge e^c \wedge e^d\right] + F_1^4\mathrm{Vol}[\Sigma_4]. \end{aligned} \tag{6.24}$$

Using (6.13) with $R = -12$, the $Spin(7)$ condition reduces to

$$\begin{aligned} \partial_u(F_1^4) &= 3F_1^2F_2, \\ \frac{1}{2}\partial_u(F_1^2F_2^2) &= \frac{1}{3}F_2^3 - F_1^2F_2. \end{aligned} \tag{6.25}$$

Defining a new coordinate x such that

$$\partial_u = \frac{3}{4}\partial_x, \tag{6.26}$$

we get

$$F_1 = x + \alpha, \tag{6.27}$$

for a constant α which may be eliminated by a shift in x. Then

$$F_2^2 = \frac{1}{x^{4/3}}\left(\beta - \frac{4}{5}x^{10/3}\right), \tag{6.28}$$

for a constant β which may be set to unity up to an overall scale in the metric. Defining a new coordinate $x^{10/3} = 5R^{10/3}/4$, up to an overall scale we obtain the $\mathcal{N}_\tau$ metric given above.

7 Conclusions and outlook

In this paper, the notion of an interpolation between Anti-de Sitter and special holonomy manifolds has been defined. The importance of this concept in the geometry of the supersymmetric AdS/CFT correspondence has been stressed. Two conjectures have been made: that all supersymmetric AdS solutions of M-/string theory admit a special holonomy interpolation, and that, with the exception of flat space, all metrics on special holonomy manifolds admitting an AdS interpolation are incomplete. For a representative sample of known supersymmetric AdS solutions of M-theory, a series of canditate incomplete special holonomy interpolations has been derived. The series of interpolations is closely related to a set of celebrated complete special holonomy metrics.

Several interesting directions for future research are suggested by the results of this paper. The geometrical question of most importance is undoubtedly the construction of an interpolating solution describing a wrapped brane, for one of the proposed interpolating pairs of this paper. Since the whole series of pairs share many common features, understanding how to do this for one of them would almost certainly facilitate the construction of an interpolating solution for all. A reasonable guess for what the boundary conditions of an interpolating solution for these pairs should be is the following. It should match on to an $\mathcal{N}_\tau$ metric at its regular degeneration point. It should also match on the AdS solution at a degeneration point of its transverse space. There is an unfixed volume modulus in all of the $\mathcal{N}_\tau$ metrics; this will be fixed, in an interpolating solution, by the global topological requirement of matching onto an AdS solution. For the AdS solutions without R-symmetry isometries, the degeneration points of the transverse space are symmetric; an interpolating solution should match on to one of them. For the AdS solutions with R-symmetry isometries, the degeneration points of the transverse space are asymmetric; in this case, it seems

plausible that an interpolating solution should match on to the AdS solution at its R-symmetry degeneration point. Understanding how this comes about, and solving the wrapped brane equations for an interpolating solution, is not just a problem in Riemannian geometry. It seems very likely that the *Lorenztian* character of an interpolating solution will enter in an essential way, with the causal structure of the interpolating solution playing a key part. This is because (at least by analogy with conical interpolations) an interpolating solution should match on to the special holonomy manifold at a spacelike infinity, and the AdS manifold at an event horizon. Of the two coordinates which play a rôle in the frame rotation underlying the relationship between the interpolating pairs of this paper, one has a finite range while the range of the other is infinite. Though they cannot really be seperated, in a rough sense the non-compact direction should determine the Lorentzian, causal structure, and the compact direction the Riemannian. A very delicate interplay between the two is required, to fulfill the appropriate Lorenztian and Riemannian boundary conditions for an interpolating solution. Understanding the geometry of the frame rotation in more depth may reveal how to linearise the wrapped brane equations, and so superimpose the interpolating pair, just as for conical interpolations. Another intriguing point about the frame rotation is that the relationship between the AdS and Minkowski frame coordinates is in every case given by the root of a polynomial. This strongly suggests some deeper underlying algebraic geometry which has not been appreciated.

Other interesting geometrical questions raised by this work include the following. For branes wrapped on Kähler cycles, there exist rich classes of AdS solutions that have not been studied here. These include AdS_5 solutions from M-fivebranes on two-cycles in three-folds [13], AdS_3 solutions from M-fivebranes on four-cycles in fourfolds [28], [30] and AdS_3 solutions from D3-branes on two-cycles in four-folds [29], [30]. It would be interesting to apply the methods of this paper to these other solutions, and so determine candidate interpolations. For the AdS-from-D3-brane solutions of [29], [30] it should be particularly feasible to construct the interpolating solutions, since in this case the four-fold geometry is essentially conical [62], [30], [59]. Also AdS_2 M-theory solutions have not been discussed in this paper at all; a rich class has recently been discovered in [30], and some older ones are to be found in [24]. Using the classification results of [63], [40], it would be interesting to determine their candidate interpolations.

It should also be possible to use the notion of an interpolating pair to construct new AdS solutions. For all cases other than Kähler cycles, to the knowledge of the author, only a single AdS solution is known to exist - the one studied in this paper. On the other hand, numerous complete cohomogeneity-one special holonomy metrics are known; for example, for G_2 and $Spin(7)$, several complete metrics, whose construction was inspired by the BSGPP metrics, were given in [52], [53]. Hyperbolic analogues of these metrics should also exist, and if so, it will almost certainly be possible to map them to new AdS solutions.

A more long-term project concerns the construction of the conformal quantum duals of the interpolating pairs. In M-theory, this problem is hampered by the notoriously intractable question of the effective field theory on the worldvolume of a stack of fivebranes (for membranes, some interesting progress on the world-volume theory, highlighting its non-associativity, has recently been made in [64]). In IIB, this is less of a problem, and it should be possible to make progress constructing the duals of wrapped D3-brane geometries, even with existing techniques.

In the geometry of wrapped brane physics, we have for so long been restricted to the near-horizon limit, the AdS geometry, that it has become commonplace to think that *only* this geometry is of relevance to investigations of the CFT. Indeed, recently it has been shown that it is in fact possible in principle to reconstruct the CFT from the near-horizon geometry alone[11] using holographic renormalisation techniques [65], [66]. However, doing this for AdS geometries of the complexity of those studied in this paper is likely to be very difficult indeed, if not impossible, in practice. And focussing on the AdS geometry alone ignores the central message of this paper: that the geometry of AdS/CFT involves, in an essential way, *both* an Anti-de Sitter and a special holonomy manifold. It is also possible, as a matter of principle, to construct the CFT dual from the geometry of the special holonomy manifold alone. It is worth recalling that this is how the quiver gauge theory duals of the $Y^{p,q}$ manifolds were in fact constructed [16], [17]; as, indeed, was $\mathcal{N} = 4$ super Yang Mills itself in this context [1]. Knowing both members of an interpolating pair means that CFT construction techniques can be brought to bear on both geometries; knowing both significantly enriches our understanding of the correspondence.

8 Acknowledgements

I would like to express my gratitude to Jerome Gauntlett for ongoing discussions, collaboration and debate; and also to Dina Daskalopoulou, for general inspiration. This work was supported by EPSRC.

References

[1] J. M. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity", Adv.Theor.Math.Phys. **2** (1998) 231-252; Int.J.Theor.Phys. **38** (1999) 1113-1133, hep-th/9711200.

[2] N. Beisert, "The S-Matrix of AdS/CFT and Yangian symmetry", PoS (Solvay) **002** (2007), arXiv:0705.0321.

[11] I thank Marika Taylor for pointing this out to me.

[3] K. Zarembo, "Semiclassical Bethe ansatz and AdS/CFT", Comptes Rendus Physique **5** (2004) 1081, Fortsch.Phys. **53** (2005) 647.

[4] J. Maldacena and C. Nuñez, "Towards the large N limit of pure N=1 super Yang Mills", Phys.Rev.Lett. **86** (2001) 588-591, hep-th/0008001.

[5] U. Gursoy and E. Kiritsis, "Exploring improved holographic theories for QCD: Part 1", arXiv:0707.1324.

[6] U. Gursoy, E. Kiritsis and F. Nitti, "Exploring improved holographic theories for QCD: part II", arXiv:0707.1349.

[7] S. A. Hartnoll and C. P. Herzog, "Ohm's law at strong coupling: S duality and the cyclotron resonance", arXiv:0706.3228.

[8] S. A. Hartnoll, P. K. Kovtun, M. Mueller and S. Sachdev, "Theory of the Nernst effect near quantum phase transitions in condensed matter, and in dyonic black holes", arXiv:0706.3215.

[9] S. A. Hartnoll and P. Kovtun, "Hall conductivity from dyonic black holes", arXiv:0704.1160.

[10] D. Mateos, R. C. Myers and R. M. Thomson, "Holographic viscosity of fundamental matter", Phys. Rev. Lett **98** (2007), 101601, hep-th/0610184.

[11] I. R. Klebanov and M. J. Strassler, "Supergravity and a Confining Gauge Theory: Duality Cascades and χSB-Resolution of Naked Singularities", JHEP **0008** (2000) 052, hep-th/0007191.

[12] I. R. Klebanov and E. Witten, "Superconformal Field Theory on Threebranes at a Calabi-Yau Singularity", Nucl.Phys. B **536** (1998) 199-218, hep-th/9807080.

[13] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Supersymmetric AdS5 solutions of M-theory", Class.Quant.Grav. **21** (2004) 4335-4366, hep-th/0402153.

[14] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Sasaki-Einstein Metrics on S^2xS^3", Adv.Theor.Math.Phys. **8** (2004) 711-734, hep-th/0403002.

[15] B. Feng, A. Hanany and Y.-H. He, "D-Brane Gauge Theories from Toric Singularities and Toric Duality", Nucl.Phys. B **595** (2001) 165-200, hep-th/0003085.

[16] D. Martelli and J. Sparks, "Toric Geometry, Sasaki-Einstein Manifolds and a New Infinite Class of AdS/CFT Duals", Commun.Math.Phys. **262** (2006) 51-89, hep-th/0411238.

[17] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, "An Infinite Family of Superconformal Quiver Gauge Theories with Sasaki-Einstein Duals", JHEP **0506** (2005) 064, hep-th/0411264.

[18] K. Intrilligator and B. Wecht, "The Exact Superconformal R-Symmetry Maximizes a", Nucl.Phys. B**667** (2003) 183-200, hep-th/0304128.

[19] D. Martelli, J. Sparks and S.-T. Yau, "The Geometric Dual of a-maximisation for Toric Sasaki-Einstein Manifolds", Commun.Math.Phys. **268** (2006) 39-65, hep-th/0503183.

[20] M. J. Duff, G. W. Gibbons and P. K. Townsend, "Macroscopic superstrings as interpolating solitons", Phys. Lett. B **332** (1994) 321-328, hep-th/9405124.

[21] J. Maldacena and C. Nuñez, "Supergravity description of field theories on curved manifolds and a no go theorem", Int.J.Mod.Phys. **A 16** (2001) 822-855, hep-th/0007018.

[22] B. S. Acharya, J. P. Gauntlett and N. Kim, "Fivebranes wrapped on associative three-cycles", Phys. Rev. D **63** (2001) 106003, hep-th/0011190.

[23] M. Pernici and E. Sezgin, "Spontaneous compactification of seven-dimensional supergravity theories", Class. Quant. Grav. **2** (1985), 673.

[24] J. P. Gauntlett, N. Kim and D. Waldram, "M-fivebranes wrapped on supersymmetric cycles", Phys. Rev. D **63** (2001) 126001, hep-th/0012195.

[25] J. P. Gauntlett and N. Kim, "M-fivebranes wrapped on supersymmetric cycles II", Phys. Rev. D**65** (2002) 086003, hep-th/0109039.

[26] H. Nieder and Y. Oz, "Supergravity and D-branes wrapping supersymmetric cycles", JHEP **0103** (2001) 008, hep-th/0011288.

[27] M. Naka, "Various wrapped branes from gauged supergravities," hep-th/0206141.

[28] J. P. Gauntlett, O. A. P. Mac Conamhna, T. Mateos and D. Waldram, "New supersymmetric AdS3 solutions", Phys.Rev. D **74** (2006) 106007, hep-th/0608055.

[29] J. P. Gauntlett, O. A. P. Mac Conamhna, T. Mateos and D. Waldram, "Supersymmetric AdS3 solutions of type IIB supergravity", Phys.Rev.Lett **97** (2006) 171601, hep-th/0606221.

[30] J. P. Gauntlett, N. Kim and D. Waldram, "Supersymmetric AdS(3), AdS(2) and bubble solutions," JHEP **0704** (2007) 005, hep-th/0612253.

[31] O. A. P. Mac Conamhna, "Inverting geometric transitions: explicit Calabi-Yau metrics for the Maldacena-Nuñez solutions", arXiv:0706.1795.

[32] A. Fayyazuddin and D. J. Smith, "Localized intersections of M5-branes and four-dimensional superconformal field theories", JHEP **9904** (1999) 030, [hep-th/9902210].

[33] H. Cho, M. Emam, D. Kastor and J. Traschen, "Calibrations and Fayyazuddin-Smith Spacetimes", Phys.Rev. D **63** (2001) 064003, hep-th/0009062.

[34] T. Z. Husain, "That's a wrap!", JHEP **0304** (2003) 053, hep-th/0302071.

[35] B. Brinne, A. Fayyazuddin, T. Z. Husain and D. J. Smith, "$N = 1$ M5-brane geometries", JHEP **0103** (2001) 052, hep-th/0012194.

[36] A. Fayyazuddin, T. Z. Husain and I. Pappa, "The geometry of wrapped M5-branes in Calabi-Yau 2-folds", hep-th/0509018.

[37] D. Martelli and J. Sparks, "G-structures, fluxes and calibrations in M-theory", Phys. Rev. D **68** (2003) 085014, hep-th/0306225.

[38] T. Z. Husain, "M2-branes wrapped on holomorphic curves", JHEP **0312** (2003) 037, hep-th/0211030.

[39] J. P. Gauntlett, O. A. P. Mac Conamhna, T. Mateos and D. Waldram, "AdS spacetimes from wrapped M5 branes," JHEP **0611** (2006) 053, hep-th/0605146.

[40] O. A. P. Mac Conamhna and E. Ó Colgáin, "Supersymmetric wrapped membranes, AdS(2) spaces, and bubbling geometries", JHEP **0703** (2007) 115, hep-th/0612196.

[41] P. Figueras, O. A. P. Mac Conamhna and E. Ó Colgáin, "Global geometry of the supersymmetric AdS_3/CFT_2 correspondence in M-theory", Phys. Rev. D **76** (2007) 046007, hep-th/0703275.

[42] T. Eguchi and A. J. Hanson, "Asymptotically flat self-dual solutions to Euclidean gravity", Phys. Lett. B **74**, 249 (1978).

[43] R. L. Bryant and S. Salamon, "On the construction of some complete metrics with exceptional holonomy", Duke Math. J. **58**, 829 (1989).

[44] G. W. Gibbons, D. N. Page and C. N. Pope, "Einstein metrics on S^3, R^3 and R^4 bundles", Commun. Math. Phys. **127**, 529 (1990).

[45] P. Candelas and X. C. de la Ossa, "Comments on Conifolds," Nucl. Phys. B **342** (1990) 246.

[46] L. A. Pando Zayas and A. A. Tseytlin, “3-branes on resolved conifold,” JHEP **0011** (2000) 028, hep-th/0010088.

[47] G. Papadopoulos and A. A. Tseytlin, “Complex geometry of conifolds and 5-brane wrapped on 2-sphere”, Class. Quant. Grav. **18** (2001) 1333, hep-th/0012034.

[48] M. B. Stenzel, “Ricci-flat metrics on the complexification of a compact rank one symmetric space”, Manuscripta Mathematica **80**, 151 (1993).

[49] M. Cvetic, G. W. Gibbons, H. Lü and C. N. Pope, “Ricci-flat metrics, harmonic forms and brane resolutions”, Commun.Math.Phys. **232** (2003) 457, hep-th/0012011.

[50] M. Cvetic, G. W. Gibbons, H. Lü and C. N. Pope, “Special holonomy spaces and M-theory”, hep-th/0206154.

[51] K. Ohta and T. Yokono, “Deformation of conifold and intersecting branes”, JHEP **0002** (2000) 023, hep-th/9912266.

[52] M. Cvetic, G. W. Gibbons, H. Lü and C. N. Pope, “Cohomogeneity one manifolds of Spin(7) and G_2 holonomy”, Phys. Rev. D**65** (2002) 106004, hep-th/0108245.

[53] S. Gukov and J. Sparks, “M-theory on Spin(7) manifolds”, Nucl. Phys. B **625** (2002) 3, hep-th/0109025.

[54] E. Calabi, “Métriques Kählériennes et fibrés holomorphe”, Ann. Scient. École Norm. Sup., **12**, 269 (1979).

[55] A. Dancer and A. Swann, “Hyperkähler metrics of cohomogeneity one”, J. Geometry and Physics **21**, 218 (1997).

[56] M. Cvetic, G. W. Gibbons, H. Lü and C. N. Pope, “Hyper-Kähler Calabi Metrics, L^2 harmonic forms, resolved M2-branes, and AdS_4/CFT_3 correspondence”, Nucl. Phys. B **617** (2001) 151, hep-th/0102185.

[57] O. A. P. Mac Conamhna, “The geometry of extended null supersymmetry in M-theory”, Phys. Rev. D **73** (2006) 045012, hep-th/0505230.

[58] A. Lukas and P. Saffin, “M-theory compactification, fluxes and AdS_4”, Phys. Rev. D **71** (2005) 046005, hep-th/0403235.

[59] J. P. Gauntlett and O. A. P. Mac Conamhna, “AdS spacetimes from wrapped D3 branes”, arXiv:hep-th/0707.3105.

[60] O. A. P. Mac Conamhna, "Eight-manifolds with G-structure in eleven dimensional supergravity", Phys.Rev. D **72** (2005) 086007, hep-th/0504028.

[61] J. P. Gauntlett, J. B. Gutowski and S. Pakis, "The Geometry of D=11 Null Killing Spinors", JHEP **0312** (2003) 049, hep-th/0311112.

[62] N. Kim, "AdS(3) solutions of IIB supergravity from D3-branes," JHEP **0601** (2006) 094, hep-th/0511029.

[63] N. Kim and J. D. Park, "Comments on AdS(2) solutions of D = 11 supergravity," JHEP **0609** (2006) 041, hep-th/0607093.

[64] J. Bagger and N. Lambert, "Modelling multiple M2's", Phys. Rev. D **75**, (2007) 045020, hep-th/0611108.

[65] K. Skenderis, "Lecture notes on holographic renormalisation", Class. Quant. Grav. **19** (2002) 5849, hep-th/0209067.

[66] K. Skenderis and M. Taylor, "Kaluza-Klein holography", JHEP **0605** (2006) 057, hep-th/0603016.